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Effects of system parameters on the optimal cost and policy in a class of multi-dimensional queueing control problems

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We consider a class of Markov Decision Processes frequently employed to model queueing and inventory control problems. For these problems, we explore how changes in different system input parameters (transition rates, costs, discount rates etc.) affect the optimal cost and the optimal policy when the state space of the problem is multi-dimensional. To address a large class of problems, we introduce two generic dynamic programming operators to model different types of controlled events. For these operators, we derive sufficient conditions to propagate monotonicity and supermodularity properties of the value function. These properties allow to predict how changes in system input parameters affect the optimal cost and policy. Finally, we explore the case when several parameters are changed at the same time.

Key words: Markov decision process, optimal policy, sensitivity analysis, event based dynamic programing.

1. Introduction

Many interesting queueing and inventory control problems can be modeled by continuous-time Markov Decision Processes. A lot of research effort has gone into the investigation of optimal policy structure in such problems. These problems are especially challenging when the state space of the problem is multi-dimensional as in the case of queueing systems with multiple queues or inventory systems with multiple products for example. Even though each new model is different and presents new challenges, a powerful approach called event-based dynamic programming proposed by Koole (1998, 2006) provides a way of establishing results on the optimal policy structure for a certain class of models in a unified manner.

Our main objective in this paper is to further explore optimal policy structure in multi-dimensional queueing and inventory control problems. In particular, we investigate how the optimal cost (or reward) and the optimal policy change when problem input parameters change. The input parameters in question are the transition rates (or probabilities) governing the controlled Markov chain and the financial parameters such as costs, rewards or discount rates. We are seeking answers to questions such as: how does the optimal cost and the optimal policy change when the customer arrival rate or the waiting cost increases in a controlled queueing system? To answer such questions, we construct a systematic approach that first builds on earlier results to infer optimal policy structure and then develops a methodology to explore the effects of varying input parameters.

Characterizing and understanding optimal policy structure in queueing and inventory control has received considerable attention. One reason for this emphasis is that understanding the theoretically optimal policy for a simplified model of reality is useful in designing near optimal working policies. This is certainly the case if the optimal policy is not completely defined by a few parameters (such as a few threshold values) as in most multi-dimensional problems. In this case, the policy designer would greatly benefit from information about how the working policy parameters should be adjusted and the cost implications of such adjustments when input conditions change. Our methodology addresses this question at a fairly general level.

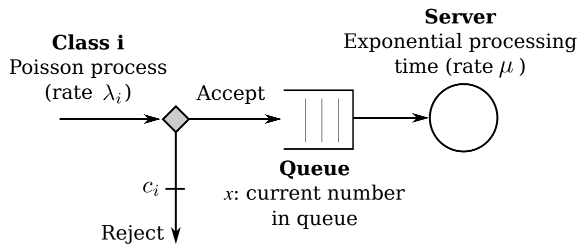
To summarize, in this paper we address the problem of how the changes in input parameters of a multi-dimensional Markov Decision Process impact the optimal value function in a relatively model-free setting. This is in contrast with most structural results reported in the literature for value functions of stochastic dynamic programs which are model specific and usually consider monotonicity only in the state variables but rarely investigate monotonicity properties in terms of the input parameters. Our contributions can be summarized as follows: First, we propose a general set of conditions that ensure different monotonicity properties (increasingness, convexity, supermodularity) of the value function in terms of the input parameters. This enables comparative statics type results on how the optimal policy and the optimal cost change as certain parameters change. To our knowledge, this is new in the multi-dimensional setting. In addition, some of the results (convexity/concavity of the value function with respect to an input parameter, compensation between operators) are novel even for single dimensional problems which are relatively well studied. Second, because monotonicity properties for multi-dimensional problems are very tedious to check on all possible commonly used individual dynamic programming operators and under combinations of different state space constraints (limited buffers, problem specific state space boundaries etc.), we introduce two generic operators that capture different types of controlled and uncontrolled transitions in the state space. These operators include as special cases several well-known queueing and inventory control operators from the literature. We present a complete monotonicity investigation for these two operators which leads to a general set of conditions that that can be expressed in a concise manner and can be adapted to specific models. This is useful for recovering earlier individual results within a general framework but more importantly provides a recipe for establishing monotonicity results for new problems that can be modeled by the generic operators.

In what follows, we present two examples that will be used throughout the paper to illustrate our approach and results. The first example explores a well-known admission control problem where the state space is single dimensional. Here, we can contrast the new approach with the existing methodology and demonstrate how some of the earlier results can be complemented. The second example is based on the make-to-stock version of a tandem queueing system. In this case, the state space is multi-dimensional and the optimal policy structure is more complicated.

Example 1: Admission control. Consider the following admission control problem with n classes of customers, adapted from Stidham (1985) and illustrated in Figure 1. Customers of class$-i$ arrive according to a Poisson process with rate λ_i and are either accepted or rejected at cost c_i. Once accepted, customers are not differentiated and the service time of the single server is exponentially distributed with rate μ. The state is the number $x \in \mathbb{Z}^+$ of customers in the system. The waiting cost is h per customer per unit of time. The objective is to choose the optimal admission control policy in order to minimize the expected discounted/average rejection and holding cost over an infinite horizon (with discount rate η). Stidham (1985) proves that the optimal policy is a threshold policy where a class-i arrival is accepted if and only if $x < t_i$. In addition $t_i \le t_j$ if $c_i \le c_j$. Çil et al. (2009) establish that t_i is increasing in μ and decreasing in λ. We will complement these results by showing that the thresholds are increasing with h and c_i and decreasing with η (see Section 6).

For this example, we also explore the effect of a parameter change on the optimal cost. For a single class of customers, Figure 2a shows that the optimal cost is increasing and concave in the holding cost h. We also observe that the optimal cost is linear in each interval where the optimal threshold is constant. We will prove these results, among others, in Section 6. Figure 2b provides

Figure 1 Admission control model with n classes of customers



an example where the optimal cost is decreasing in μ but is neither concave nor convex. However the optimal cost is convex in each interval where the optimal threshold is constant.

Table 1 explores the case when multiple input parameters change simultaneously and illustrates what we call compensation between several parameters. We consider three instances where we vary simultaneously the arrival rates of three classes of customers in such a way that the sum of arrival rates remains equal to 1.8. We observe that the optimal thresholds of Instance 2 are smaller than in Instance 1. However we can not order the thresholds of Instance 1 and Instance 3 in a similar manner. In Section 7, we will provide conditions under which we can predict the increase or decrease of the optimal thresholds and costs.

Table 1 Compensation ($\mu = 1$, $h = 1$, $c_1 = 5$, $c_2 = 10$, and $c_3 = 15$)

Instance	λ_1	λ_2	λ_3	t_1	t_2	t_3	Optimal cost
1	0.6	0.6	0.6	1	2	5	8.98
2	0.1	0.7	1	0	1	4	12.1
3	0.1	1.6	0.1	0	3	7	10.6

Example 2 : Tandem queue. This second example is to illustrate the effect of changing parameters on the optimal policy in a two-dimensional problem. The make-to-stock tandem queue model of Veatch and Wein (1992, 1994) is illustrated in Figure 3. Servers M_i produce items one by one, with exponentially distributed processing times (rate μ_i). Produced items at server i are held in buffer B_i. Demand, if not immediately satisfied, is backlogged in buffer B_d. The state of the system is described by (x_1, x_2) with x_1 the number of work-in-process products in B_1 and x_2 the number of serviceable products in B_2 minus the number of backlogged demand in B_d. The system incurs a holding cost h_i per unit of time and unit of product in buffer B_i and a backorder cost b per unit of time and unit of waiting demand. The objective is to minimize the expected discounted/average cost over an infinite horizon (with discount rate η). Veatch and Wein (1992) prove that the optimal production policy is a state dependent base stock policy defined by two switching curves: Produce at M_i iff $x_2 < s_i(x_1)$, for $i = 1, 2$. Please note that unlike a simple optimal threshold policy as in the previous admission control example, describing the optimal policy requires specifying complete functions for the two switching curves. The design of a practical near optimal policy is therefore an issue. Veatch and Wein (1992) report that two-stage Kanban policies described by two parameters perform quite well. In designing such practical policies, it is useful to understand how the policy parameters should be adjusted when problem inputs change.

Figures 4a and 4b show the influence of the demand rate λ and the service rate μ_2 on the optimal switching curves. We observe that λ has a monotonic effect on the switching curves. The switching curve s_1 (resp. s_2) for $\lambda = 1$ is systematically below the one for $\lambda = 1.1$. We will prove in Section 6 that this result holds in general. On the other hand, we observe that μ_2 has a non-monotonic effect on the switching curve s_1: The curve for $\mu_2 = 1.2$ crosses the curve for $\mu_2 = 2$. This implies that different input parameters may have different monotonicity consequences.

The rest of the paper is organized as follows. Next section reviews the literature and our contributions. Section 3 presents the class of problems and operators under consideration. Section 4 introduces several properties of value functions and state spaces. Section 5 presents our approach and main results to study the effect of changing parameters on the optimal cost and policy. Section 6 applies our results to the admission control and tandem queue problems. Section 7 exhibits compensation phenomena when several parameters are changed simultaneously.

2. Literature review

Structure of the optimal policy. In a number of queueing control problems, the optimal policy can be described by thresholds, switching curves, or hyperplanes. Several papers develop general approaches for deriving structural properties of the optimal policy (Weber and Stidham 1987, Veatch and Wein 1992, Smith and McCardle 2002, Zhuang and Li 2012). In particular Koole (1998, 2006) presents the so-called event-based dynamic programming framework to study queueing control problems. In this framework, an operator is associated to each type of event (demand arrival, end of service, processor failure, etc) and can be studied individually. To characterize the structure of the optimal policy, some properties of the value function such as monotonicity, convexity/concavity and supermodularity/submodularity are needed. If each individual event operator propagates a desired property, then the optimal value function, which is a composition of different individual operators, will also possess this property.

Effect of system parameters on the optimal cost. Many papers investigate numerically the effect of some problem input parameters on the optimal cost and the optimal policy in specific queueing control problems. There is a rich literature on how average performance measures change in terms of the input parameters for uncontrolled queueing systems. However, very few papers investigate this question in the context of controlled queueing systems from a theoretical point of view. Müller

Figure 2 Effect of parameters on the optimal cost

(a) Concavity and piecewise linearity in h ($n=1$, $\lambda_1=0.6$, $c_1=5$, $\mu=1$, $\eta=0$)

(b) Piecewise convexity in μ ($n=1$, $\lambda_1=0.6$, $c_1=5$, $h=1$, $\eta=0$).

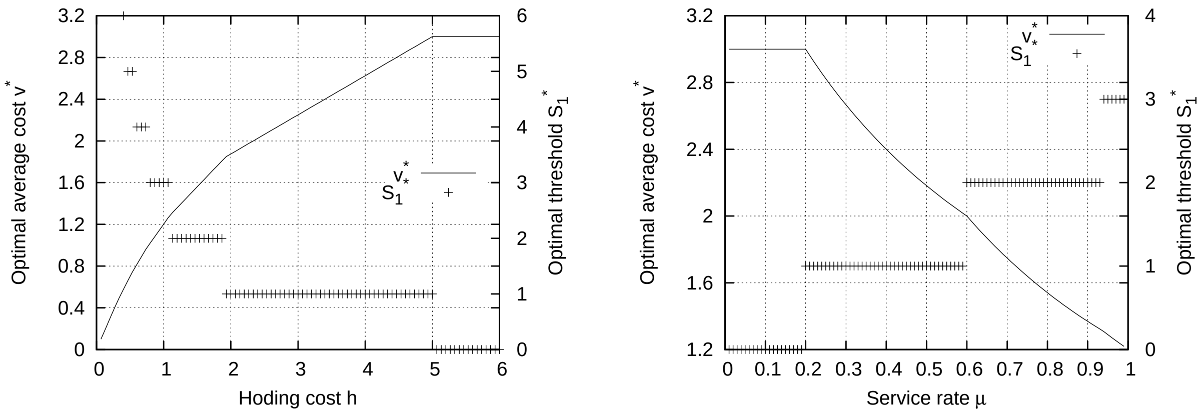


Figure 3 Tandem make-to-stock queue model

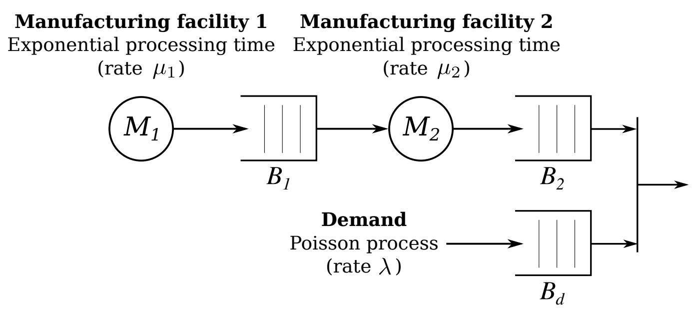
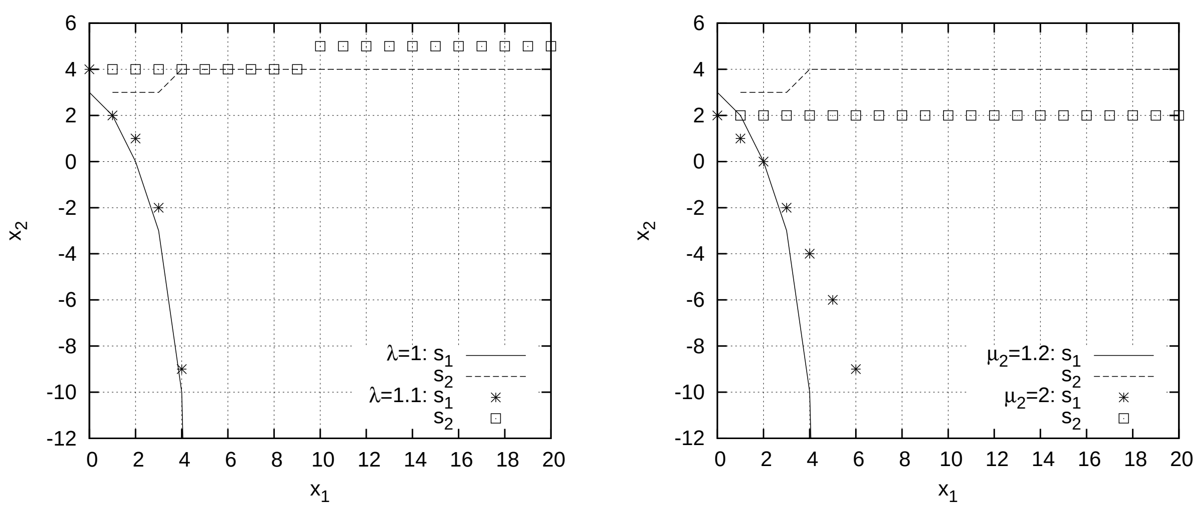


Figure 4 Effect of parameters on the optimal policy

(a) Monotonic effect of λ ($\mu_1=2$, $\mu_2=1.2$, $h_1=1$, $h_2=2$, $b=4$, $\eta=0.1$)

(b) Non monotonic effect of μ_2 ($\lambda=1$, $\mu_1=2$, $h_1=1$, $h_2=2$, $b=4$, $\eta=0.1$)



(1997) compares the optimal value function of discrete time Markov decision processes that differ only in their transition probabilities. His method requires establishing some complex and restrictive stochastic orders and exploiting some (a priori) known properties of the value function. Koole (2006) employs the event-based dynamic programming framework to address the problem but the relevant monotonicity and convexity results of the optimal value function are obtained for operators with no decision (arrival, departure, etc).

Effect of system parameters on the optimal policy. The implications on the optimal policy of changes in input parameters is less studied. Some papers such as Ku and Jordan (1997), Gans and Savin (2007), Aktaran-Kalaycı and Ayhan (2009) investigate the policy effects of changing input parameters in specific queueing control examples. Zhuang and Li (2012) propose a method based on the general property of multimodularity to establish structural results on the optimal policy for a class of problems. For a specific example, they show that multimodularity also enables obtaining monotonicity results on the parameters of the optimal policy with respect to the input parameters. In contrast, Çil et al. (2009) develop a general approach using the framework of event-based dynamic programming to systematically study the effects of changing input parameters. However, their analysis is mostly restricted to problems where the state space is single dimensional and the optimal policy can be described by thresholds. Their results have been employed in several recent papers for different applications (see e.g. Aydin et al. (2009), Zerhouni et al. (2013), Benjaafar et al. (2010), Satir et al. (2012), Ozkan et al. (2013)).

Contributions with respect to the literature. We explore the effects of changing input parameters in a general class of queueing or inventory control problems. This is in contrast with the rich literature on queueing and inventory control that only explores monotonicity properties of the value function in terms of the state variables. It is also much more general in scope than the problem specific analysis with respect to input parameters as in Ku and Jordan (1997), Gans and Savin (2007), Aktaran-Kalaycı and Ayhan (2009). Koole (1998, 2006) propose a quite general approach to study monotonicity in the state variables which includes multi-dimensional models but report some relatively limited results on monotonicity in the input parameters only for uncontrolled models (i.e. queueing systems with no dynamic decisions). Çil et al. (2009) obtain monotonicity results in the input parameters for both controlled and uncontrolled models but their analysis is limited to single-dimensional models. Our level of generality in modeling is close to Koole (1998) in that we can address a fairly large class of multi-dimensional models. On the other hand, our approach is different because unlike Koole (1998) and Çil et al. (2009) we do not study monotonicity properties individually for a long list of individual dynamic programming operators but focus on two generic operators that cover most of that long list (and some other models that may not be part of the list). We are then able to obtain general conditions for monotonicity for these two operators that can be adapted to all special cases. To our knowledge, the results pertaining to the effects of input parameters are new. In addition, we think the level of generality is useful even when monotonicity in the state variables is sought for new models and/or under arbitrary state space boundary structures.

3. The operators

The main notations used in this work are summarized in Table 2.

Consider a continuous-time MDP with the objective to minimize the expected discounted cost over an infinite horizon with discount rate η. Our results can be easily adapted to maximization problems, finite horizon problems or average cost problems.

The state is an m_1-dimensional vector $\mathbf{s_1} \in \mathcal{S}_1 \subset \mathbb{Z}^{m_1}$ where $\mathbb{Z}$ is the set of all integers. We assume that the system parameters (transition rates, costs, discount rate) can be summarized in a m_2-dimensional vector $\mathbf{s_2} \in \mathcal{S}_2 \subset \mathbb{R}^{m_2}$ where $\mathbb{R}$ is the set of real numbers. We can aggregate $\mathbf{s_1}$ and $\mathbf{s_2}$ in an (m_1+m_2)-dimensional vector $\mathbf{x} = (\mathbf{s_1}, \mathbf{s_2}) \in \mathcal{X} = \mathcal{S}_1 \times \mathcal{S}_2$. In the rest of the paper, vector $\mathbf{x}$ will be referred to as the *system state* while $\mathbf{s_1}$ will be simply called the *state*. The set $\mathcal{X}$ will be referred to as the *system state space*. We illustrate below the system state on our two examples.

Admission control $\mathbf{x} = (\underbrace{x}_{\mathbf{s_1}}, \underbrace{\mu, \lambda_1, \cdots, \lambda_n, h, c_1, \cdots, c_n, \eta}_{\mathbf{s_2}})$.

Tandem queue $\mathbf{x} = (\underbrace{x_1, x_2}_{\mathbf{s_1}}, \underbrace{\mu_1, \mu_2, \lambda, h_1, h_2, b, \eta}_{\mathbf{s_2}})$.

Let $v^\star(\mathbf{x}) = v^\star(\mathbf{s_1}, \mathbf{s_2})$ be the optimal expected discounted cost over an infinite horizon when the initial state is $\mathbf{s_1}$ and system parameters are given by $\mathbf{s_2}$.

$$v^\star(\mathbf{x}) = \boldsymbol{\mathcal{M}} v^\star(\mathbf{x}), \tag{1}$$

where $\boldsymbol{\mathcal{M}}$ is the optimal operator.

We assume that the optimal operator can be decomposed as a convex combination of individual operators corresponding to each event, using the well-known method of uniformization (Lippman 1975):

$$\boldsymbol{\mathcal{M}} v(\mathbf{x}) = \frac{1}{\eta + \sum_{i=0}^{l} p_i} \left(\mathcal{H}(\mathbf{x}) + \sum_{i=1}^{l} p_i \mathcal{O}_i v(\mathbf{x}) + p_0 v(\mathbf{x}) \right). \tag{2}$$

The operator $\mathcal{H}$ is a cost rate function which does not depend on decisions. The operator $\mathcal{O}_i$ is associated to the i-th type of event which occurs with rate p_i. The last term $p_0 v$ corresponds to a fictitious event which occurs with rate p_0 and affects neither the state nor the cost of the system. This term will be useful to compare systems with different event rates or discount rates in order to keep constant the quantity $\eta + \sum_{i=0}^{l} p_i$. For instance, if the arrival rate increases by ϵ, the fictitious

$\mathbb{Z}$	Set of integers
$\mathbb{Z}^+$	Set of positive integers
$\mathbb{R}$	Real numbers
$\mathbf{s_1}$	State ($\mathbf{s_1} \in \mathcal{S}_1$)
$\mathbf{s_2}$	Vector of system parameters ($\mathbf{s_2} \in \mathcal{S}_2$)
$\mathbf{x}$	System state ($\mathbf{x} = (\mathbf{s_1}, \mathbf{s_2}) \in \mathcal{X}$)
$\boldsymbol{\epsilon}$	System parameter perturbation (same dimension as $\mathbf{x}$, can change only $\mathbf{s_2}$)
$\mathbf{a}, \mathbf{b}, \mathbf{d}$	State translation (same dimension as $\mathbf{x}$, can change only $\mathbf{s_1}$)
$\boldsymbol{\alpha}, \boldsymbol{\beta}$	System state translation (same dimension as $\mathbf{x}$, can change $\mathbf{s_1}$ and $\mathbf{s_2}$)
$\mathbf{y}$	$= \mathbf{x} + \mathbf{b}$
$\mathbf{e}_i$	$= (0, \dots, 0, 1, 0, \dots, 0)$ with "1" in ith position
ϵ_p	Perturbation of parameter p
$v(\mathbf{x})$	Value function when the initial state is $\mathbf{x}$
$v^\star(\mathbf{x})$	Optimal value function when the initial state is $\mathbf{x}$
η	Discount rate
$\boldsymbol{\mathcal{M}}$	Optimal operator
$\mathcal{O}$	Event operator
$\mathcal{O}_i$	Operator associated to the i-th type of event
p_i	Occurrence rate of event associated to operator $\mathcal{O}_i$
$\mathcal{H}$	Cost function
$\mathcal{T}$	Translation operator: $\mathcal{T}v(\mathbf{x}) = \begin{cases} v(\mathbf{y}+\mathbf{a}) + c_a & \text{if } \mathbf{y}+\mathbf{a} \in \mathcal{X}, \\ v(\mathbf{y}) + c_r & \text{otherwise.} \end{cases}$
$\mathcal{C}$	Choice operator: $\mathcal{C}v(\mathbf{x}) = \begin{cases} \min\{v(\mathbf{y}+\mathbf{a}) + c_a, v(\mathbf{y}) + c_b\} & \text{if } \mathbf{y}+\mathbf{a} \in \mathcal{X}, \\ v(\mathbf{y}) + c_r, & \text{otherwise.} \end{cases}$
c_d	$= c_a - c_b$ (cost difference in the choice operator $\mathcal{C}$)
$\mathtt{P}$	v is positive
$\mathtt{N}$	v is negative
$\mathtt{I}_{\boldsymbol{\alpha}}$	v is increasing in direction $\boldsymbol{\alpha}$
$\mathtt{D}_{\boldsymbol{\alpha}}$	v is decreasing in direction $\boldsymbol{\alpha}$
$\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$	v is supermodular in directions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$
$\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\alpha}}$	v is convex in direction $\boldsymbol{\alpha}$
$\mathtt{S}_{\boldsymbol{\alpha},-\boldsymbol{\alpha}}$	v is concave in direction $\boldsymbol{\alpha}$
$\mathtt{S}^{ub}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$	v is submodular in directions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$
$\Delta_{\boldsymbol{\alpha}} v(\mathbf{x})$	Differentiation in direction $\boldsymbol{\alpha}$ ($= v(\mathbf{x}+\boldsymbol{\alpha}) - v(\mathbf{x})$)
$\Omega_{\mathcal{O}} v(\mathbf{x})$	Marginal cost of operator $\mathcal{O}$ ($= \mathcal{O}v(\mathbf{x}) - v(\mathbf{x})$)
$\mathtt{PM}(\mathcal{O})$	Operator $\mathcal{O}$ has a positive marginal cost
$\mathtt{NM}(\mathcal{O})$	Operator $\mathcal{O}$ has a negative marginal cost
$\mathtt{IM}_{\mathbf{d}}(\mathcal{O})$	Operator $\mathcal{O}$ has an increasing marginal cost in direction $\mathbf{d}$
$\mathtt{DM}_{\mathbf{d}}(\mathcal{O})$	Operator $\mathcal{O}$ has a decreasing marginal cost in direction $\mathbf{d}$
$\mathtt{R}_{\mathbf{a}_1,\dots,\mathbf{a}_l}(\mathbf{b})$	Property of the state space $\mathcal{X}$ (see Definition 1)
$\lvert\text{assertion}\rvert$	$= true$ iff the assertion is true
$\wedge$	Logical conjonction "and"
$\vee$	Logical disjonction "or"

Table 2 Main notations

rate decreases by ϵ. Without loss of generality, we set $\eta + \sum_{i=0}^{l} p_i = 1$ which is equivalent to set a time unit.

In this paper, we consider two new operators that generalize several operators from the literature. Let $\mathbf{y} = \mathbf{x} + \mathbf{b}$. The *translation operator* $\mathcal{T}$ and the *choice operator* $\mathcal{C}$ are defined as

$$\mathcal{T}v(\mathbf{x}) = \begin{cases} v(\mathbf{y}+\mathbf{a}) + c_a & \text{if } \mathbf{y}+\mathbf{a} \in \mathcal{X}, \\ v(\mathbf{y}) + c_r & \text{otherwise.} \end{cases}$$

$$\mathcal{C}v(\mathbf{x}) = \begin{cases} \min\{v(\mathbf{y}+\mathbf{a}) + c_a, v(\mathbf{y}) + c_b\} & \text{if } \mathbf{y}+\mathbf{a} \in \mathcal{X}, \\ v(\mathbf{y}) + c_r, & \text{otherwise.} \end{cases}$$

In the definition of $\mathcal{T}$ and $\mathcal{C}$, we implicitly assume that if $\mathbf{x} \in \mathcal{X}$ then $\mathbf{y} \in \mathcal{X}$. The decision in the choice operator depends on the sign of the cost difference $c_d = c_a - c_b$. Operator $\mathcal{C}$ will reduce to operator $\mathcal{T}$ if the optimal decision is always to move to state $\mathbf{y}+\mathbf{a}$ (for instance if c_b goes to infinity).

Table 3 illustrates how several operators from the literature (Koole 1998, 2006, Çil et al. 2009) can be seen as special cases of the translation and choice operators. In this table and the rest of the paper, $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the unit vector in direction i (the "1" is in i^{th} position).

We now provide some examples of operators that are not treated by our generic operators. The choice operator does not address situations with more than two choices, typically the Movable Server Departure operator

$$T_{MSD}v(x) = \min_{i=1,\cdots,n} \{v(x - e_i)\},$$

when the number of choices is strictly larger than 2 ($n > 2$). The translation operator can not either address operators with state-dependent service rate, typically the Parallel Server Departure operator

$$T_{PSD}v(x) = \begin{cases} \frac{x_i}{n} v(x - e_i) + \frac{n - x_i}{n} v(x) & \text{if } x_i < n, \\ v(x - e_i) & \text{otherwise,} \end{cases}$$

when the number of servers is strictly larger than 1 ($n > 1$).

Finally, we define two last operators $\Delta_{\boldsymbol{\alpha}}$ and $\Omega_{\mathcal{O}}$ with $\boldsymbol{\alpha}$ a translation of the system state and $\mathcal{O}$ an ad-hoc operator:

$$\Delta_{\boldsymbol{\alpha}} v(\mathbf{x}) = v(\mathbf{x} + \boldsymbol{\alpha}) - v(\mathbf{x}),$$

$$\Omega_{\mathcal{O}} v(\mathbf{x}) = \mathcal{O}v(\mathbf{x}) - v(\mathbf{x}).$$

The quantity $\Omega_{\mathcal{O}} v(\mathbf{x})$ represents the marginal cost associated to the decision made by operator $\mathcal{O}$.

Table 3 Some operators from the literature (Koole 1998, 2006, Çil et al. 2009) as special cases of the translation and choice operators. Unless specified, we set $\mathbf{b}=\mathbf{0}$, $c_a=c_b=c_r=c_a=0$ and $\mathcal{S}_1=(\mathbb{Z}^+)^{m_1}$

Name	Operator from the literature	With choice and translation operators
Arrival	$T_{A(i)}v(\mathbf{x})=v(\mathbf{x}+\mathbf{e}_i)$	$\mathcal{T}v(\mathbf{x})$ with $\mathbf{a}=\mathbf{e}_i$
Departure	$T_{D(i)}v(\mathbf{x})=v((\mathbf{x}-\mathbf{e}_i)^+)$	$\mathcal{T}v(\mathbf{x})$ with $\mathbf{a}=-\mathbf{e}_i$
Parallel departure	$T_{PD}v(\mathbf{x})=\sum_k \gamma_k v((\mathbf{x}-\mathbf{e}_k)^+)$	$\sum_k \gamma_k \mathcal{T}_k v(\mathbf{x})$ with $\mathbf{a}_k=-\mathbf{e}_k$
Tandem server	$T_{T(i,j)}v(\mathbf{x})=v((\mathbf{x}-\mathbf{e}_i+\mathbf{e}_j)^+)$	$\mathcal{T}v(\mathbf{x})$ with $\mathbf{a}=\mathbf{e}_j-\mathbf{e}_i$
Controlled arrival	$T_{CA(i)}v(\mathbf{x})=\min\{v(\mathbf{x});v(\mathbf{x}+\mathbf{e}_i)+c\}$	$\mathcal{C}v(\mathbf{x})$ with $\mathbf{a}=\mathbf{e}_i,\ c_a=c$
Controlled arrival as fork	$T_{CAF}v(\mathbf{x})=\min\{v(\mathbf{x});v(\mathbf{x}+\sum_k \mathbf{e}_k)+c\}$	$\mathcal{C}v(\mathbf{x})$ with $\mathbf{a}=\sum_k \mathbf{e}_k,\ c_a=c$
Routing	$T_{R(i,j)}v(\mathbf{x})=\min_{k\in\{i,j\}} v(\mathbf{x}+\mathbf{e}_k)+c^k$	$\mathcal{C}v(\mathbf{x})$ with $\mathbf{a}=\mathbf{e}_j-\mathbf{e}_i$, $\mathbf{b}=\mathbf{e}_i,\ c_a=c^j,\ c_b=c^i$
Batch arrival	$T_{BA(i)}v(\mathbf{x})=\min_{0\geq j\geq B} v(\mathbf{x}+j\mathbf{e}_i)+jc$	$\mathcal{C}_1(\mathcal{C}_2(\dots(\mathcal{C}_B v)\dots)(\mathbf{x})$ with $\mathbf{a}=\mathbf{e}_i,\ c_a=c,\ B>0$
Controlled departure	$T_{CD(i)}v(\mathbf{x})=\begin{cases}\min\{v(\mathbf{x}),v(\mathbf{x}-\mathbf{e}_i)+c\} \text{ if } x_i>0,\\ v(\mathbf{x}) \text{ otherwise,}\end{cases}$	$\mathcal{C}v(\mathbf{x})$ with $\mathbf{a}=-\mathbf{e}_i,\ c_a=c$
Controlled tandem server	$T_{CT(i,j)}v(\mathbf{x})=\begin{cases}\min\{v(\mathbf{x}),v(\mathbf{x}-\mathbf{e}_i+\mathbf{e}_j)+c\}\\ \quad \text{if } x_i>0,\\ v(\mathbf{x}) \text{ otherwise.}\end{cases}$	$\mathcal{C}v(\mathbf{x})$ with $\mathbf{a}=\mathbf{e}_j-\mathbf{e}_i,\ c_a=c$

4. Value function and state space properties

In the following definitions, $v\geq 0$ means that for all $\mathbf{s_1}$, $v(\mathbf{s_1},\mathbf{s_2})\geq 0$ (the value of $\mathbf{s_2}$ will be clear from the context). The word increasing (resp. decreasing, positive, negative) is used for non-decreasing (resp. non-increasing, non-negative, non-positive).

We first define some properties of a value function:

$$\mathtt{P}: v\geq 0 \quad \text{(positive)}\ ,$$
$$\mathtt{N}: v\leq 0 \quad \text{(negative)}\ ,$$
$$\mathtt{I}_{\boldsymbol{\alpha}}: \Delta_{\boldsymbol{\alpha}} v\geq 0 \quad \text{(increasing)}\ ,$$
$$\mathtt{D}_{\boldsymbol{\alpha}}: \Delta_{\boldsymbol{\alpha}} v\leq 0 \quad \text{(decreasing)}\ ,$$
$$\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}: \Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}} v\geq 0 \quad \text{(supermodularity)}\ ,$$
$$\mathtt{S}^{ub}_{\boldsymbol{\alpha},\boldsymbol{\beta}}: \Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}} v\leq 0 \quad \text{(submodularity)}\ .$$

We can see $\mathtt{P}$, $\mathtt{N}$, $\mathtt{I}_{\boldsymbol{\alpha}}$, $\mathtt{D}_{\boldsymbol{\alpha}}$, $\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\mathtt{S}^{ub}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ as Boolean variables. For instance, $\mathtt{I}_{\boldsymbol{\alpha}}$ is true if the assertion "$\Delta_{\boldsymbol{\alpha}} v \geq 0$" is true. We will use notation $\wedge$ (resp. $\vee$) for Boolean operator "and" (resp. "or"). Moreover $|a|$ will be a Boolean variable which is true when the assertion "a" is *true*. Thus $\mathtt{I}_{\boldsymbol{\alpha}} = |\Delta_{\boldsymbol{\alpha}} v \geq 0|$. By convention, the "and" operator $\wedge$ takes precedence over the "or" operator $\vee$.

We show in A.1 that :

i) $\mathtt{I}_{\boldsymbol{\alpha}} = \mathtt{D}_{-\boldsymbol{\alpha}}$,

ii) $\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \mathtt{S}^{ub}_{-\boldsymbol{\alpha},\boldsymbol{\beta}} = \mathtt{S}^{ub}_{\boldsymbol{\alpha},-\boldsymbol{\beta}} = \mathtt{S}_{-\boldsymbol{\alpha},-\boldsymbol{\beta}}$,

iii) $\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \wedge \mathtt{S}_{\boldsymbol{\gamma},\boldsymbol{\beta}}$ implies $\mathtt{S}_{\boldsymbol{\alpha}+\boldsymbol{\gamma},\boldsymbol{\beta}}$,

For instance, property iii) means that if v is $\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\mathtt{S}_{\boldsymbol{\gamma},\boldsymbol{\beta}}$, then v is $\mathtt{S}_{\boldsymbol{\alpha}+\boldsymbol{\gamma},\boldsymbol{\beta}}$.

We also define some properties related to the marginal cost operator $\Omega_{\mathcal{O}}$:

$$\mathtt{PM}(\mathcal{O}) : \Omega_{\mathcal{O}} v \geq 0 \text{ (positive marginal cost) ,}$$

$$\mathtt{NM}(\mathcal{O}) : \Omega_{\mathcal{O}} v \leq 0 \text{ (negative marginal cost) ,}$$

$$\mathtt{IM}_{\boldsymbol{\alpha}}(\mathcal{O}) : \Delta_{\boldsymbol{\alpha}} \Omega_{\mathcal{O}} v \geq 0 \text{ (increasing marginal cost) ,}$$

$$\mathtt{DM}_{\boldsymbol{\alpha}}(\mathcal{O}) : \Delta_{\boldsymbol{\alpha}} \Omega_{\mathcal{O}} v \leq 0 \text{ (decreasing marginal cost) .}$$

Again we can see $\mathtt{PM}(\mathcal{O})$, $\mathtt{NM}(\mathcal{O})$, $\mathtt{IM}_{\boldsymbol{\alpha}}(\mathcal{O})$ and $\mathtt{DM}_{\boldsymbol{\alpha}}(\mathcal{O})$ as Boolean variables. We have $\mathtt{IM}_{\boldsymbol{\alpha}}(\mathcal{O}) = \mathtt{DM}_{-\boldsymbol{\alpha}}(\mathcal{O})$.

The form of the state space is also important when characterizing the optimal policy. The simplest case is when the state space is infinite in all directions. However results can be derived for other state spaces that can be described by the following property, illustrated in Figure 5.

Definition 1. A state space $\mathcal{X}$ is $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_l}(\mathbf{b})$ if for all $\mathbf{x}$ such that $\{\mathbf{x}, \mathbf{x}+\mathbf{a}_1, \ldots, \mathbf{x}+\mathbf{a}_l\} \subset \mathcal{X}$, then $\mathbf{x}+\mathbf{b} \in \mathcal{X}$.

Again, we can see $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_l}(\mathbf{b})$ as a Boolean variable which is true when $\mathcal{X}$ is $\mathtt{R}_{\mathbf{a}_1,\ldots}(\mathbf{b})$. One can easily check the following properties (see A.2 for a proof for vi) :

i) $\mathtt{R}(\mathbf{b})$: the set $\mathcal{X}$ is invariant by translation $\mathbf{b}$

ii) $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_l}(\mathbf{0}) = true$

iii) $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_l}(\mathbf{a}_i) = true$

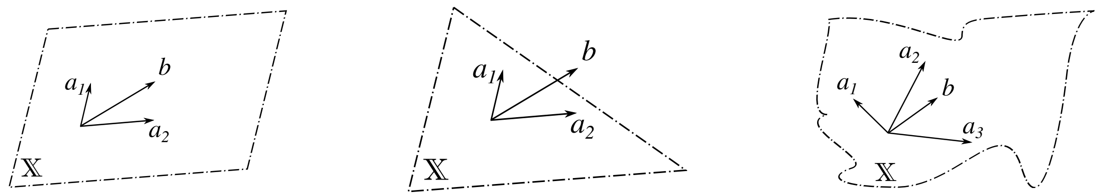
iv) $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_i,\mathbf{a}_j,\ldots,\mathbf{a}_l}(\mathbf{b}) = \mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_j,\mathbf{a}_i,\ldots,\mathbf{a}_l}(\mathbf{b})$

v) $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_{l-1}}(\mathbf{b})$ implies $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_l}(\mathbf{b})$

vi) $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_l}(\mathbf{b}) = \mathtt{R}_{\mathbf{a}_1-\mathbf{a}_l,\ldots,\mathbf{a}_{l-1}-\mathbf{a}_l,-\mathbf{a}_l}(\mathbf{b}-\mathbf{a}_l)$

Figure 5 Illustration of $\mathtt{R}$ properties on different system state spaces

(a) $\mathcal{X}$ is $\mathtt{R}_{\mathbf{a_1},\mathbf{a_2}}(\mathbf{b})$ (b) $\mathcal{X}$ is not $\mathtt{R}_{\mathbf{a_1},\mathbf{a_2}}(\mathbf{b})$ (c) $\mathcal{X}$ is $\mathtt{R}_{\mathbf{a_1},\mathbf{a_2},\mathbf{a_3}}(\mathbf{b})$



5. Properties of the operators

A system parameter perturbation $\boldsymbol{\epsilon} = (0, \ldots, 0, \epsilon_1, \ldots, \epsilon_{m_2})$, with $\epsilon_i \in \mathbb{R}$, translates the system state from $\mathbf{x}$ to $\mathbf{x} + \boldsymbol{\epsilon}$. Such a translation can modify the transition rates p_i, the discount rate η and the costs c_a, c_r, c_b of the operators. However a perturbation $\boldsymbol{\epsilon}$ does not change the state, the state space or the action space. Let $\epsilon_{p_i}, \epsilon_\eta, \epsilon_{c_a}, \epsilon_{c_r}, \epsilon_{c_b}$ denote the perturbation for parameters p_i, η, c_a, c_r, c_b.

In this section, we provide sufficient conditions such that the optimal value function is positive/negative, increasing/decreasing in the direction $\boldsymbol{\epsilon}$, convex/concave in the direction $\boldsymbol{\epsilon}$. We also provide sufficient conditions such that the optimal switching curves, if any, increase or decrease with $\boldsymbol{\epsilon}$.

Table 4 summarizes our results for the operators. A detailed proof of each result can be found in Appendix B for the translation operator and in Appendix C for the choice operator. Section 6 will illustrate how to use these results for the admission control problem and the tandem queue problem.

5.1. Sign of the optimal cost

To study the effect of the discount rate on the optimal value function, we will need results on the sign of the optimal value function. This section is trivial but has the merit to introduce the approach and notations in a simple way.

The optimal value function $v^\star$ is positive (resp. negative) if the optimal operator $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{P}$ (resp. $\mathtt{N}$) (i.e. if v is $\mathtt{P}$ then $\boldsymbol{\mathcal{M}}v$ is $\mathtt{P}$). From (2), we have

Proposition 1. *$\boldsymbol{\mathcal{M}}$ propagates $\mathtt{P}$ if the following Boolean variable is true.*

$$|\mathcal{H} \geq 0| \bigwedge_{i=1}^{l} |\mathcal{O}_i \textit{ propagate } \mathtt{P}|$$

$\boldsymbol{\mathcal{M}}$ propagates $\mathtt{N}$ if the following Boolean variable is true.

$$|\mathcal{H} \leq 0| \bigwedge_{i=1}^{l} |\mathcal{O}_i \textit{ propagate } \mathtt{N}|$$

In order to apply Proposition 1, we need to prove that each individual operator $\mathcal{O}_i$ propagates $\mathtt{P}$ (or $\mathtt{N}$). Table 4 (cells 1 to 4) provide sufficient conditions for the translation and the choice operators to propagate $\mathtt{P}$ (or $\mathtt{N}$). These results are trivial and simply state that an operator propagates $\mathtt{P}$ (or $\mathtt{N}$) if all its costs are positive (or negative). The proof for each cell is in B and C.

We point out that the translation operator is a special case of choice operator when c_b tends to infinity (i.e. $\Delta_{\mathbf{a}} v + c_a - c_b \leq 0$). However, we kept the two operators to facilitate the model and the use of the results. It should be noted that our results for choice operator are sufficient conditions, and that we simplify the results assuming that $\Delta_{\mathbf{a}} v + c_a - c_b$ sometimes positive and sometimes negative (i.e. $\exists \mathbf{x}_1, \mathbf{x}_2$ such that $\Delta_{\mathbf{a}} v(\mathbf{x}_1) + \geq c_b - c_a$ and $\Delta_{\mathbf{a}} v(\mathbf{x}_2) \leq c_b - c_a$). That is why the results on left column is not a particular case of the right column when c_b tends to infinity. It is up to the user of our results to consider translation operator if $\Delta_{\mathbf{a}} v + c_a - c_b$ always positive or negative.

5.2. Monotonicity of the optimal cost

To study the monotonicity of the optimal value function in direction $\boldsymbol{\epsilon}$, we can limit our analysis to $\mathtt{I}_{\boldsymbol{\epsilon}}$ as $\mathtt{D}_{\boldsymbol{\epsilon}} = \mathtt{I}_{-\boldsymbol{\epsilon}}$. The optimal value function $v^\star$ is $\mathtt{I}_{\boldsymbol{\epsilon}}$ if the optimal operator $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$.

From (2), we have

$$\boldsymbol{\mathcal{M}} v(\mathbf{x}+\boldsymbol{\epsilon}) = \mathcal{H}(\mathbf{x}+\boldsymbol{\epsilon}) + \textstyle\sum_{i=1}^{l} (p_i + \epsilon_{p_i}) \mathcal{O}_i v(\mathbf{x}+\boldsymbol{\epsilon}) + (p_0 - \epsilon_\eta - \sum_{i=1}^{l} \epsilon_{p_i}) v(\mathbf{x}+\boldsymbol{\epsilon}). \tag{3}$$

From (2) and (3), it follows that

$$\Delta_{\boldsymbol{\epsilon}} \boldsymbol{\mathcal{M}} v(\mathbf{x}) = \begin{pmatrix} \Delta_{\boldsymbol{\epsilon}} \mathcal{H}(\mathbf{x}) \\ + p_0 \Delta_{\boldsymbol{\epsilon}} v(\mathbf{x}) \\ + \sum_{i=1}^{l} p_i \Delta_{\boldsymbol{\epsilon}} \mathcal{O}_i v(\mathbf{x}) \\ + \sum_{i=1}^{l} \epsilon_{p_i} \Omega_{\mathcal{O}_i} v(\mathbf{x}+\boldsymbol{\epsilon}) \\ - \epsilon_\eta v(\mathbf{x}+\boldsymbol{\epsilon}) \end{pmatrix}. \tag{4}$$

This quantity is positive if each line is positive. The sign of the first line depends on the problem under consideration. The second line is positive if v is $\mathtt{I}_{\boldsymbol{\epsilon}}$. As $p_i > 0$, the third line is positive if each operator $\mathcal{O}_i$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$. The fourth line is positive if ϵ_{p_i} and the marginal cost $\Omega_{\mathcal{O}_i} v$ have the same sign, or if $\epsilon_{p_i} = 0$. Finally the last line is positive if v and ϵ_η have opposite signs, or if $\epsilon_\eta = 0$.

Using Boolean notations, this leads to the following proposition which provides sufficient conditions for the optimal operator to propagate $\mathtt{I}_{\boldsymbol{\epsilon}}$.

PROPOSITION 2. $\boldsymbol{\mathcal{M}}$ *propagates* $\mathtt{I}_{\boldsymbol{\epsilon}}$ *if the following Boolean variable is true.*

$$|\Delta_{\boldsymbol{\epsilon}} \mathcal{H} \geq 0| \bigwedge_{i=1}^{l} \left[\begin{matrix} |\mathcal{O}_i \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \wedge \begin{pmatrix} |\epsilon_{p_i} < 0| \wedge |\Omega_{\mathcal{O}_i} v \leq 0| \\ \vee |\epsilon_{p_i} > 0| \wedge |\Omega_{\mathcal{O}_i} v \geq 0| \\ \vee |\epsilon_{p_i} = 0| \end{pmatrix} \end{matrix} \right] \bigwedge \begin{pmatrix} |\epsilon_\eta < 0| \wedge |v \text{ is } \mathtt{P}| \\ \vee |\epsilon_\eta > 0| \wedge |v \text{ is } \mathtt{N}| \\ \vee |\epsilon_\eta = 0| \end{pmatrix}.$$

Table 4 Sufficient conditions for properties of the operators

	Translation operator ($\mathcal{O}=\mathcal{T}$)	Choice operator ($\mathcal{O}=\mathcal{C}$)
$\mathcal{O}$ propagates $\mathtt{P}$	1) $|c_a \geq 0| \wedge \begin{pmatrix} |c_r \geq 0| \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$	2) $|c_a \geq 0| \wedge |c_b \geq 0| \wedge \begin{pmatrix} |c_r \geq 0| \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$
$\mathcal{O}$ propagates $\mathtt{N}$	3) $|c_a \leq 0| \wedge \begin{pmatrix} |c_r \leq 0| \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$	4) $|c_a \leq 0| \wedge |c_b \leq 0| \wedge \begin{pmatrix} |c_r \leq 0| \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$
$\mathcal{O}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$	5) $|\epsilon_{c_a} \geq 0| \wedge \begin{pmatrix} |\epsilon_{c_r} \geq 0| \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$	6) $|\epsilon_{c_a} \geq 0| \wedge |\epsilon_{c_b} \geq 0| \wedge \begin{pmatrix} |\epsilon_{c_r} \geq 0| \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$
$\mathcal{O}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$	7) *true*	8) $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_d} = 0|$
$\mathcal{O}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$	9) *true*	10) $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_d} \geq 0| \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_d} \leq 0|$
$\mathcal{O}$ propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$	11) $\begin{pmatrix} \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_r} \geq \epsilon_{c_a}| \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \end{pmatrix} \wedge \begin{pmatrix} \mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_a} \geq \epsilon_{c_r}| \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$	12) $\begin{pmatrix} \mathtt{S}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_d} \leq 0| \\ \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_d} \geq 0| \\ \vee \mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}} \wedge \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}) \wedge |\epsilon_{c_d} = 0| \end{pmatrix} \wedge \begin{pmatrix} \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_r} \geq \epsilon_{c_a}| \wedge |\epsilon_{c_r} \geq \epsilon_{c_b}| \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \end{pmatrix} \wedge \begin{pmatrix} \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}+\mathbf{a}} \wedge |\epsilon_{c_a} \geq \epsilon_{c_r}| \wedge |\epsilon_{c_b} \geq \epsilon_{c_r}| \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$
Positive marginal cost: $\Omega_{\mathcal{O}} v \geq 0$	13) $|\Delta_{\mathbf{a}+\mathbf{b}} v \geq -c_a| \wedge \begin{pmatrix} |\Delta_{\mathbf{b}} v \geq -c_r| \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$	14) $|\Delta_{\mathbf{b}} v \geq -c_b| \wedge |\Delta_{\mathbf{a}+\mathbf{b}} v \geq -c_a| \wedge \begin{pmatrix} |\Delta_{\mathbf{b}} v \geq -c_r| \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$
Negative marginal cost: $\Omega_{\mathcal{O}} v \leq 0$	15) $|\Delta_{\mathbf{a}+\mathbf{b}} v \leq -c_a| \wedge \begin{pmatrix} |\Delta_{\mathbf{b}} v \leq -c_r| \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$	16) $\begin{pmatrix} |\Delta_{\mathbf{b}} v \leq -c_b| \\ \vee |\Delta_{\mathbf{a}+\mathbf{b}} v \leq -c_a| \end{pmatrix} \wedge \begin{pmatrix} |\Delta_{\mathbf{b}} v \leq -c_r| \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$
Increasing marginal cost : $\Delta_{\boldsymbol{\epsilon}} \Omega_{\mathcal{O}} v \geq 0$	17) $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{a}+\mathbf{b}} \wedge |\epsilon_{c_a} \geq 0| \wedge \begin{pmatrix} \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge |\epsilon_{c_r} \geq 0| \\ \vee \mathtt{R}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$	18) $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{a}} \wedge |\epsilon_{c_a} \geq 0| \wedge |\epsilon_{c_b} \geq 0| \wedge \begin{pmatrix} \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge |\epsilon_{c_r} \geq 0| \\ \vee \mathtt{R}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$
Increasing marginal cost: $\Delta_{\mathbf{d}} \Omega_{\mathcal{O}} v \geq 0$	19) $\mathtt{S}_{\mathbf{d},\mathbf{a}+\mathbf{b}} \wedge \begin{pmatrix} \mathtt{S}_{\mathbf{d},\mathbf{b}} \\ \vee \mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \\ \vee \mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix} \wedge \begin{pmatrix} |\Delta_{\mathbf{a}} v \leq c_r - c_a| \wedge [\mathtt{S}_{\mathbf{d},\mathbf{b}} \vee \mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}}] \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \end{pmatrix} \wedge \begin{pmatrix} |\Delta_{\mathbf{a}} v \geq c_r - c_a| \wedge [\mathtt{S}_{\mathbf{d},\mathbf{b}} \vee \mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}}] \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$	20) $\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge \mathtt{S}_{\mathbf{d},\mathbf{a}} \wedge \begin{pmatrix} |c_r \geq \max\{-c_b, c_b\}| \\ \vee \mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}} \wedge |\Delta_{\mathbf{a}} v \leq c_r - c_a| \wedge |c_r \geq c_b| \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \end{pmatrix} \wedge \begin{pmatrix} |c_b \geq c_r| \wedge |\Delta_{\mathbf{a}} v \geq c_r - c_a| \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$

To apply Proposition 2, we need to prove that each individual operator $\mathcal{O}_i$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ (see cells 5 and 6 of Table 4 for sufficient conditions). For each operator $\mathcal{O}_i$ such that $\epsilon_{p_i} \neq 0$, we also need to show that the marginal cost is either positive or negative (see cells 13 and 14 for sufficient conditions).

The formalism used in Proposition 2 and Table 4 represent in a compact way the effect of many parameters for a large class of operators. When considering the effect of a single parameter Proposition 2 and Table 4 reduce drastically. For example consider the effect of an increase $\epsilon_{p_1} > 0$ of parameter p_1 on the optimal value function. From Proposition 2, $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{I}_{\epsilon_{p_1}}$ if each $\mathcal{O}_i$ propagates $\mathtt{I}_{\epsilon_{p_1}}$ and if $\Omega_{\mathcal{O}_1} v \geq 0$. From cells 5 and 6, the translation and choice operators propagates $\mathtt{I}_{\epsilon_{p_1}}$ since $\epsilon_{c_a} = \epsilon_{c_r} = \epsilon_{c_b} = 0$. Remains to check the positivity of the marginal cost.

5.3. Convexity or concavity of the optimal cost

A value function v is convex in direction $\boldsymbol{\epsilon}$ if it is $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$, i.e. $\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} v(\mathbf{x}) \geq 0$. It is concave in direction $\boldsymbol{\epsilon}$ if it is $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$.

We want to find sufficient conditions such that operator $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ or $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$. From (2), we have

$$\begin{aligned} \boldsymbol{\mathcal{M}} v(\mathbf{x}+2\boldsymbol{\epsilon}) &= \mathcal{H}(\mathbf{x}+2\boldsymbol{\epsilon}) + \textstyle\sum_{i=1}^{l} (p_i + 2\epsilon_{p_i}) \mathcal{O}_i v(\mathbf{x}+2\boldsymbol{\epsilon}) \\ &\quad + (p_0 - 2\epsilon_\eta - 2\textstyle\sum_{i=1}^{l} \epsilon_{p_i}) v(\mathbf{x}+2\boldsymbol{\epsilon}). \end{aligned} \tag{5}$$

From (2), (3) and (5), it follows that

$$\begin{aligned} \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} \boldsymbol{\mathcal{M}} v(\mathbf{x}) &= \boldsymbol{\mathcal{M}} v(\mathbf{x}+2\boldsymbol{\epsilon}) - 2\boldsymbol{\mathcal{M}} v(\mathbf{x}+\boldsymbol{\epsilon}) + \boldsymbol{\mathcal{M}} v(\mathbf{x}) \\ &= \begin{pmatrix} \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} \mathcal{H}(\mathbf{x}) \\ +p_0 \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} v(\mathbf{x}) \\ +\sum_{i=1}^{l} p_i \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} \mathcal{O}_i v(\mathbf{x}) \\ +2\sum_{i=1}^{l} \epsilon_{p_i} \Delta_{\boldsymbol{\epsilon}} \Omega_{\mathcal{O}_i} v(\mathbf{x}+\boldsymbol{\epsilon}) \\ -2\epsilon_\eta \Delta_{\boldsymbol{\epsilon}} v(\mathbf{x}+\boldsymbol{\epsilon}) \end{pmatrix}. \end{aligned} \tag{6}$$

This quantity is positive (respectively negative) if each line is positive (respectively negative). This leads to the following proposition.

Proposition 3. *$\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ if the following Boolean variable is true.*

$$|\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} \mathcal{H} \geq 0| \bigwedge_{i=1}^{l} \left[\begin{array}{c} |\mathcal{O}_i \ \textit{propagates}\ \mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}| \\ \wedge \begin{pmatrix} |\epsilon_{p_i} < 0| \wedge |\Delta_{\boldsymbol{\epsilon}} \Omega_{\mathcal{O}_i} v \leq 0| \\ \vee |\epsilon_{p_i} > 0| \wedge |\Delta_{\boldsymbol{\epsilon}} \Omega_{\mathcal{O}_i} v \geq 0| \\ \vee |\epsilon_{p_i} = 0| \end{pmatrix} \end{array} \right] \bigwedge \begin{pmatrix} |\epsilon_\eta < 0| \wedge |v \ \textit{is}\ \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \vee |\epsilon_\eta > 0| \wedge |v \ \textit{is}\ \mathtt{I}_{-\boldsymbol{\epsilon}}| \\ \vee |\epsilon_\eta = 0| \end{pmatrix}.$$

$\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ if the following Boolean variable is true.

$$|\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} \mathcal{H} \leq 0| \bigwedge_{i=1}^{l} \left[\begin{array}{c} |\mathcal{O}_i \ \textit{propagates}\ \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| \\ \wedge \begin{pmatrix} |\epsilon_{p_i} > 0| \wedge |\Delta_{\boldsymbol{\epsilon}} \Omega_{\mathcal{O}_i} v \leq 0| \\ \vee |\epsilon_{p_i} < 0| \wedge |\Delta_{\boldsymbol{\epsilon}} \Omega_{\mathcal{O}_i} v \geq 0| \\ \vee |\epsilon_{p_i} = 0| \end{pmatrix} \end{array} \right] \bigwedge \begin{pmatrix} |\epsilon_\eta > 0| \wedge |v \ \textit{is}\ \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \vee |\epsilon_\eta < 0| \wedge |v \ \textit{is}\ \mathtt{I}_{-\boldsymbol{\epsilon}}| \\ \vee |\epsilon_\eta = 0| \end{pmatrix}.$$

In order to apply Proposition 3, we need to prove that each individual operator $\mathcal{O}_i$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ or $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ (see cells 7 to 10 for sufficient conditions). For each operator $\mathcal{O}_i$ such that $\epsilon_{p_i} \neq 0$, we also need to show that the marginal cost is either increasing or decreasing in the direction $\boldsymbol{\epsilon}$ (see cells 17 to 18 for sufficient conditions).

5.4. Monotonicity of the optimal policy

In this section, we study the effect of a system parameter perturbation $\boldsymbol{\epsilon}$ on the optimal policy. For the choice operator $\mathcal{C}$, the decision depends on the sign of $\Delta_{\mathbf{a}} v(\mathbf{x}) + c_d$. If $\Delta_{\mathbf{a}} v(\mathbf{x}) + c_d \geq 0$, it is optimal to stay in state $\mathbf{x}$. Otherwise, it is optimal to go in state $\mathbf{x}+\mathbf{a}$. The sign of $\Delta_{\boldsymbol{\epsilon}}(\Delta_{\mathbf{a}} v + c_d) = \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{a}} v + \epsilon_{c_d}$ will provide an indication on how the optimal policy evolves with $\boldsymbol{\epsilon}$.

Our objective is to find sufficient conditions to have $\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{a}} v + \epsilon_{c_d}$ positive. If $\epsilon_{c_d} \geq 0$, it is sufficient to show that $\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{a}} v \geq 0$ (i.e. v is $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}$). When ϵ_{c_d} is negative, we can not conclude. Sufficient conditions to have $\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{a}} v + \epsilon_{c_d}$ negative can be easily deduced by noticing that $\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} = \mathtt{S}_{\mathbf{a},-\boldsymbol{\epsilon}}$.

Let $\mathbf{d}$ be a vector in $\mathcal{X}$ that translates the state but does not change the system parameters. From (4), we have

$$\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}\boldsymbol{\mathcal{M}}v(\mathbf{x}) = \left(\begin{array}{c} \Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}\mathcal{H}(\mathbf{x}) \\ +p_0\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}} v(\mathbf{x}) \\ +\sum_{i=1}^{l} p_i \Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}\mathcal{O}_i v(\mathbf{x}) \\ +\sum_{i=1}^{l} \epsilon_{p_i}\Delta_{\mathbf{d}}\Omega_{\mathcal{O}_i} v(\mathbf{x}+\boldsymbol{\epsilon}) \\ -\epsilon_\eta \Delta_{\mathbf{d}} v(\mathbf{x}+\boldsymbol{\epsilon}) \end{array} \right). \tag{7}$$

This quantity is positive if each line is positive . This leads to the following proposition.

Proposition 4. *$\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ if the following Boolean variable is true.*

$$|\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}\mathcal{H} \geq 0| \bigwedge_{i=1}^{l} \left[\begin{array}{c} |\mathcal{O}_i \textit{ propagates } \mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}| \\ \wedge \left(\begin{array}{c} |\epsilon_{p_i} < 0| \wedge |\Delta_{\mathbf{d}}\Omega_{\mathcal{O}_i} v \leq 0| \\ \vee |\epsilon_{p_i} > 0| \wedge |\Delta_{\mathbf{d}}\Omega_{\mathcal{O}_i} v \geq 0| \\ \vee |\epsilon_{p_i} = 0| \end{array} \right) \end{array} \right] \bigwedge \left(\begin{array}{c} |\epsilon_\eta < 0| \wedge |v \textit{ is } \mathtt{I}_{\mathbf{d}}| \\ \vee |\epsilon_\eta > 0| \wedge |v \textit{ is } \mathtt{I}_{-\mathbf{d}}| \\ \vee |\epsilon_\eta = 0| \end{array} \right).$$

In order to apply Proposition 4, we need to prove that each individual operator $\mathcal{O}_i$ propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ (see cells 11 and 12 for sufficient conditions). For each operator $\mathcal{O}_i$ such that $\epsilon_{p_i} \neq 0$, we also need to show that the marginal cost is either increasing or decreasing in the direction $\mathbf{d}$ (see cells 19 and 20 for sufficient conditions).

Propositions 1, 2, 3, 4 and Table 4 present an approach to check desired structural properties for any event-based dynamic program consisting of translation and choice type operators with the appropriate cost/reward parameters, transition directions and state space restrictions. On the other hand, a large number of queueing/inventory control problems are modeled by relatively few standard operators. For these most commonly used operators, we provide a detailed set of sufficient conditions in Appendix F. These results should further facilitate applying the results of this section.

6. Illustration of results

We illustrate in this section how to apply our results to the admission control problem and the tandem queue problem, that have been introduced in Section 1. The general outline of the approach for any other problem is likely to be similar. We first identify the dynamic programming operators and the appropriate cost parameters and transition directions. We then use Propositions 1, 2, 3, 4 and Table 4 to check desired properties for the relevant operators. For the operators used in these two problems, this task is facilitated by the explicit results provided in Appendix F.

6.1. Admission control problem

The optimality equations for the admission control are

$$\begin{aligned}
\boldsymbol{\mathcal{M}}v &= \mathcal{H} + \mu\mathcal{O}_0 v + \textstyle\sum_{i=1}^n \lambda_i \mathcal{O}_i v + p_0 v, \\
\mathcal{H}(\mathbf{x}) &= hx, \\
\mathcal{O}_0 v(\mathbf{x}) &= v[(\mathbf{x}-\mathbf{e}_1)^+] = \mathcal{T} v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a} = -\mathbf{e}_1, \mathbf{b} = \mathbf{0}, \\ c_a = c_r = 0, \end{cases} \\
\mathcal{O}_i v(\mathbf{x}) &= \min(v(\mathbf{x}) + c_i, v(\mathbf{x}+\mathbf{e}_1)) = \mathcal{C} v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a} = \mathbf{e}_1, \mathbf{b} = \mathbf{0}, \\ c_b = c_i, c_a = c_r = 0, \end{cases} \quad \text{for } i = 1, \cdots, n.
\end{aligned}$$

The state space is $\mathcal{S}_1 = \mathbb{Z}^+$.

The optimal policy has been characterized in Stidham (1985). It consists of n thresholds $t_1, \cdots, t_n$. Customers of class i are accepted in the system if $x < t_i$ and rejected otherwise. If the rejection costs are ordered as $c_1 \geq \cdots \geq c_n$, then $t_1 \geq \cdots \geq t_n$. Finally the optimal value function is convex and increasing ($\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}$ and $\mathtt{I}_{\mathbf{e}_1}$). Çil et al. (2009) have shown that the optimal thresholds t_i are increasing in the service rate μ and decreasing in the arrival rates λ_i.

Using propositions 1, 2, 3, 4 and Table 4, we re-obtain these results and complement them in several directions.

Theorem 1. *In the admission control problem, the optimal value function and the optimal cost have the following properties.*

- ***Monotonicity:*** *The optimal value function is increasing in the arrival rates λ_i, the rejection costs c_i, the holding cost h and decreasing in the service rate μ and the discount rate η.*
- ***Convexity/concavity:*** *The optimal value function is concave in the holding cost h.*
- ***Monotonicity of the optimal policy:*** *The optimal thresholds t_i are decreasing in the arrival rate λ_i, the holding cost h, and increasing in the service rate μ and the discount rate η.*

Each result of Theorem 1 is proven in D.1. To illustrate the methodology, we provide below a detailed proof for the effect of the lambda rate λ_1 on the optimal cost.

Assume that v is $\mathtt{I}_{\epsilon_{\lambda_1}}$. From Proposition 2, $\boldsymbol{\mathcal{M}}v$ is $\mathtt{I}_{\epsilon_{\lambda_1}}$ if

$$|\Omega_{\mathcal{O}_1} v \geq 0 \wedge| \bigwedge_{i=0}^{n} \left| \mathcal{O}_i \text{ propagates } \mathtt{I}_{\epsilon_{\lambda_1}} \right|.$$

is true. From Cell 14 of Table 4, $\Omega_{\mathcal{O}_1} v \geq 0$ if

$$|\Delta_{\mathbf{0}} v \geq -c_i| \wedge |\Delta_{\mathbf{e}_1} v \geq 0| \wedge \begin{pmatrix} |\Delta_{\mathbf{0}} v \geq 0| \\ \vee \mathtt{R}_{\mathbf{0}}(\mathbf{e}_1) \end{pmatrix} = |\Delta_{\mathbf{e}_1} v \geq 0|$$

is true. From Cell 5 of Table 4, $\mathcal{O}_0$ propagates $\mathtt{I}_{\epsilon_{\lambda_1}}$ without condition. From Cell 6 of Table 4, $\mathcal{O}_i$ propagates $\mathtt{I}_{\epsilon_{\lambda_1}}$ without condition, for $i = 1, \cdots, n$. In the end, $\boldsymbol{\mathcal{M}} v$ is $\mathtt{I}_{\epsilon_{\lambda_1}}$ if v is $\mathtt{I}_{\epsilon_{\lambda_1}}$ and $\mathtt{I}_{\mathbf{e}_1}$.

Assume that v is $\mathtt{I}_{\epsilon_{\lambda_1}}$ and $\mathtt{I}_{\mathbf{e}_1}$, then $\boldsymbol{\mathcal{M}} v$ is $\mathtt{I}_{\epsilon_{\lambda_1}}$ from the previous paragraph. Moreover $\boldsymbol{\mathcal{M}} v$ is $\mathtt{I}_{\mathbf{e}_1}$ from Stidham (1985). By value iteration, the optimal value function $v^\star$ is $\mathtt{I}_{\epsilon_{\lambda_1}}$ and $\mathtt{I}_{\mathbf{e}_1}$.

Piecewise results. We can also derive piecewise results by looking at the effect of parameters for a set of fixed thresholds $t_1, \cdots, t_n$. If customers of class i are accepted if and only if $x_i < t_i$, operator $\mathcal{O}_i$ is replaced by the following operator that is a translation operator.

$$\tilde{\mathcal{O}}_i v(\mathbf{x}) = \begin{cases} v(\mathbf{x} + \mathbf{e}) \text{ if } s \in \{0, \ldots, t_i - 1\}, \\ v(\mathbf{x}) + c_i \text{ otherwise}, \end{cases} = \mathcal{T} v(\mathbf{x}) \text{ with } \begin{cases} \mathcal{S}_1^i = \{0, \ldots, t_i - 1\}, \\ \mathbf{a} = \mathbf{e}_1, \mathbf{b} = \mathbf{0}, \\ c_a = c_r = 0. \end{cases}$$

Using again the results of Table 4, we obtain the following theorem. The proof is in D.2.

Theorem 2. *The optimal value function is piecewise linear in the rejection costs c_i and the holding cost h and piecewise convex in the arrival rates λ_i and the service rate μ.*

This theorem is illustrated in Section 1 (see Figure 2).

6.2. Tandem queue problem

The optimality equations for the tandem queue problem are

$$\begin{aligned} \boldsymbol{\mathcal{M}} v &= \mathcal{H} + \mu_1 \mathcal{O}_1 v + \mu_2 \mathcal{O}_2 v + \lambda \mathcal{O}_3 v + p_0 v, \\ \mathcal{H}(\mathbf{x}) &= h_1 x_1 + h_2 \max\{x_2, 0\} + b \max\{-x_2, 0\}, \\ \mathcal{O}_1 v(\mathbf{x}) &= \min(v(\mathbf{x}), v(\mathbf{x} + \mathbf{e}_1)) \ = \mathcal{C} v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a} = \mathbf{e}_1, \mathbf{b} = \mathbf{0}, \\ c_a = c_r = 0, \end{cases} \\ \mathcal{O}_2 v(\mathbf{x}) &= \begin{cases} \min(v(\mathbf{x}), v(\mathbf{x} - \mathbf{e}_1 + \mathbf{e}_2)) & \text{if } x_1 > 0, \\ v(\mathbf{x}) & \text{otherwise}, \end{cases} = \mathcal{C} v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a} = \mathbf{e}_2 - \mathbf{e}_1, \mathbf{b} = \mathbf{0}, \\ c_a = c_b = c_r = 0, \end{cases} \\ \mathcal{O}_3 v(\mathbf{x}) &= v(\mathbf{x} - \mathbf{e}_2) \ = \mathcal{T} v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a} = -\mathbf{e}_2, \mathbf{b} = \mathbf{0}, \\ c_a = c_b = c_r = 0. \end{cases} \end{aligned}$$

The state space is $\mathcal{S}_1 = \mathbb{Z}^+ \times \mathbb{Z}$.

From Veatch and Wein (1992) the optimal policy consists of two switching curves: Produce at M_i iff $x_2 < s_i(x_1)$, for $i = 1, 2$. Moreover the optimal value function is $\mathtt{S}_{\mathbf{e}_1, \mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1}$, and $\mathtt{S}_{\mathbf{e}_2 - \mathbf{e}_1, \mathbf{e}_2}$. Using propositions 1, 2, 3, 4 and Table 4, we obtain the following new results for this problem.

Theorem 3. *In the tandem queue problem, the optimal value function and the optimal cost have the following properties.*

- ***Monotonicity:*** *The optimal value function is increasing in the costs h_i and b, and decreasing in the service rate μ_i and the discount rate η.*
- ***Convexity/concavity:*** *The optimal value function is concave in the costs h_2 and b.*
- ***Monotonicity of the optimal policy:*** *the optimal switching curves $s_i(x_1)$ are increasing in the demand rate λ, the backlog costs b, and decreasing in the holding cost h_2.*

Each result of Theorem 3 is proven in E. This theorem is illustrated in Section 1 for the effect of λ (see Figure 4a).

7. Compensation between operators

In Section 5, we have provided a set of sufficient conditions for different properties of the optimal value function. If each individual event operator $\mathcal{O}_i$ propagates a desired property, then the optimal operator $\boldsymbol{\mathcal{M}}$ also propagates this property. For instance, in (4) we saw that $\sum_{i=1}^{l} \epsilon_{p_i} \Omega_{\mathcal{O}_i} v \geq 0$ if $\epsilon_{p_i} \geq 0$ and $\Omega_{\mathcal{O}_i} v \geq 0$ for all i. This approach relies on the trivial property that $\sum_{i=1}^{l} u_i v_i \geq 0$ if $u_i \geq 0$ and $v_i \geq 0$ for all i.

In this section, we show that it is possible to derive another set of conditions by considering several operators simultaneously, and not individually. We will call this approach *compensation* between operators and system parameters. The following lemma provides another set of conditions to have $\sum u_i v_i \geq 0$.

Lemma 1. *Consider two sequences of real numbers (u_i) and (v_i) and set $v_0 = 0$. We have*

$$\sum_{i=1}^{l} u_i v_i = \sum_{k=1}^{l} \left[\left(\sum_{i=k}^{l} u_i \right) (v_i - v_{i-1}) \right].$$

Moreover $\sum_{i=1}^{l} u_i v_i \geq 0$ if

- *$\sum_{i=k}^{l} u_i \geq 0$ for all $k = 1, \cdots, n$*
- *and $v_i \leq v_{i+1}$ for $i = 0, \cdots, n-1$.*

Hence, in (4), we have $\sum_{i=1}^{l} \epsilon_{p_i} \Omega_{\mathcal{O}_i} v \geq 0$ if $\sum_{i=k}^{l} \epsilon_{p_i} \geq 0$ (for $k = 1, \cdots, n$) and $0 \leq \Omega_{\mathcal{O}_1} v \leq \cdots \leq \Omega_{\mathcal{O}_l} v$. Similarly in (6), we have $\sum_{i=1}^{l} \epsilon_{p_i} \Delta_{\mathbf{d}} \Omega_{\mathcal{O}_i} v \geq 0$ if $\sum_{i=k}^{l} \epsilon_{p_i} \geq 0$ for $k = 1, \cdots, n$ and $0 \leq \Delta_{\mathbf{d}} \Omega_{\mathcal{O}_1} v \leq \cdots \leq \Delta_{\mathbf{d}} \Omega_{\mathcal{O}_l} v$.

Illustration on the admission control problem

We illustrate the compensation approach on the admission control problem. We obtain the following additional results by considering together the admission control operators $\mathcal{O}_1, \cdots, \mathcal{O}_n$.

Theorem 4. *In the admission control problem, the optimal value function is increasing in $\boldsymbol{\epsilon}$ and the optimal thresholds t_i are decreasing in $\boldsymbol{\epsilon}$ if $c_1 \leq \cdots \leq c_n$, $\sum_{i=k}^{n} \epsilon_{\lambda_i} \geq 0$ (for $k = 1, \cdots, n$), $\epsilon_h \geq 0$, $\epsilon_\mu \leq 0$, $\epsilon_{c_i} \geq 0$, and $\eta \leq 0$.*

Proof of Theorem 4. We know from Table 4 that $\Omega_{\mathcal{O}_i} v$ and $\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_i} v$ are positive. Remains to show that $\Omega_{\mathcal{O}_i} v$ and $\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_i} v$ are increasing in i.

The marginal cost

$$\Omega_{\mathcal{O}_i} v(\mathbf{x}) = \min\{c_i, \Delta_{\mathbf{e}_1} v(\mathbf{x})\}$$

is increasing in i, as c_i is increasing in i and v does not depend on i.

We have $\Delta_{\mathbf{e}_1} v(\mathbf{x}) \leq \Delta_{\mathbf{e}_1} v(\mathbf{x}+\mathbf{e}_1)$ as v is convex (see Theorem 1). It follows that

$$\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_i} v(\mathbf{x}) = \begin{cases} 0 & \text{if } c_i \leq \Delta_{\mathbf{e}_1} v(\mathbf{x}) \leq \Delta_{\mathbf{e}_1} v(\mathbf{x}+\mathbf{e}_1) \\ c_i + \Delta_{\mathbf{e}_1} v(\mathbf{x}) & \text{if } \Delta_{\mathbf{e}_1} v(\mathbf{x}) \leq c_i \leq \Delta_{\mathbf{e}_1} v(\mathbf{x}+\mathbf{e}_1) \\ \Delta_{\mathbf{e}_1}\Delta_{\mathbf{e}_1} v(\mathbf{x}) & \text{if } \Delta_{\mathbf{e}_1} v(\mathbf{x}) \leq \Delta_{\mathbf{e}_1} v(\mathbf{x}+\mathbf{e}_1) \leq c_i \end{cases}$$

It follows that

$$\begin{aligned} &\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_{i+1}} v(\mathbf{x}) - \Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_i} v(\mathbf{x}) \\ &= \begin{cases} 0 & \text{if } c_i \leq c_{i+1} \leq \Delta_{\mathbf{e}_1} v(\mathbf{x}) \leq \Delta_{\mathbf{e}_1} v(\mathbf{x}+\mathbf{e}_1) \\ 0 & \text{if } c_i \leq \Delta_{\mathbf{e}_1} v(\mathbf{x}) \leq c_{i+1} \leq \Delta_{\mathbf{e}_1} v(\mathbf{x}+\mathbf{e}_1) \\ \Delta_{\mathbf{e}_1}\Delta_{\mathbf{e}_1} v(\mathbf{x}) \geq 0 & \text{if } c_i \leq \Delta_{\mathbf{e}_1} v(\mathbf{x}) \leq \Delta_{\mathbf{e}_1} v(\mathbf{x}+\mathbf{e}_1) \leq c_{i+1} \\ c_{i+1} - c_i \geq 0 & \text{if } \Delta_{\mathbf{e}_1} v(\mathbf{x}) \leq c_i \leq c_{i+1} \leq \Delta_{\mathbf{e}_1} v(\mathbf{x}+\mathbf{e}_1) \\ 0 & \text{if } \Delta_{\mathbf{e}_1} v(\mathbf{x}) \leq c_i \leq \Delta_{\mathbf{e}_1} v(\mathbf{x}+\mathbf{e}_1) \leq c_{i+1} \\ 0 & \text{if } \Delta_{\mathbf{e}_1} v(\mathbf{x}) \leq \Delta_{\mathbf{e}_1} v(\mathbf{x}+\mathbf{e}_1) \leq c_i \leq c_{i+1} \end{cases} \end{aligned}$$

is positive. □

For two classes of customers, if $c_1 \leq c_2$, Theorem 4 states that the optimal cost increases if λ_1 decreases less than λ_2 increases, which is rather intuitive. This theorem is illustrated numerically with three classes of customers in Table 1 (see Section 1).

8. Conclusion

Designing effective policies for queueing or inventory control problems requires understanding the optimal policy structure in addition to understanding how the optimal costs and the policies change if input parameters were to change. In this paper, we focus on the latter part of the problem and provide a general framework to study the effect of system parameters changes on the optimal cost and the optimal policy in multi-dimensional queueing control problems. In order to maintain modeling and analysis generality, we introduce two generic dynamic programming operators that cover many operators considered in the queuing and inventory literature. For these operators, we derive sufficient conditions on the state space and the value function to guarantee the propagation of several properties of the value function and the marginal cost (sign, monotonicity supermodularity). We also show how to apply our results on two examples for which we derive several new results.

Another contribution of the paper is to formalize a number of proofs that can be found in the literature and to investigate in a systematic way a set of necessary conditions through Boolean equations. We believe that our approach opens interesting perspectives on the automation of proofs of structural properties. A software tool would be particularly valuable for checking proofs from the literature and deriving new results that might be too complex to tackle manually. It would be also of interest to extend the results to operators that are not treated by our generic operators (e.g. movable server, state dependent event rate).

The approach and the results we present have their limitations but we think these limitations are not any more restrictive than the general state of the art in stochastic dynamic programming. The monotonicity results we present are comparative statics type results in nature. For instance, we can establish that the optimal value function is increasing and/or convex or has increasing differences in various directions with respect to some input parameter but we cannot provide numbers on the relative increase or the change (which is an interesting future research direction). Further, there are relatively few nice structural results known for value functions beyond two dimensions in queuing or inventory control and this imposes a natural constraint on the models that fall into our framework when applied to optimal policy related comparative statics. In particular, this implies that we can handle only special cases when the state space has more than two dimensions. Also consistent with the literature, we can only provide and check sufficient conditions for monotonicity. Finally, while the two operators we present cover a wide range of models, they cannot capture all individual dynamic programming operators that may arise in specific problems. An operator that is not a special case of our generic operators would then have to be studied separately along the lines of the proposed approach.

Appendix A: Properties

A.1. Properties on the value function

i) and ii) Direct consequence of the definitions of $\mathtt{I}_{\boldsymbol{\alpha}}$, $\mathtt{D}_{\boldsymbol{\alpha}}$, $\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\mathtt{S}^{ub}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

iii) We sum the two inequalities $\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}}v(\mathbf{x}+\boldsymbol{\gamma})\geq 0$ and $\Delta_{\boldsymbol{\gamma}}\Delta_{\boldsymbol{\beta}}v\geq 0$ to get $\Delta_{\boldsymbol{\alpha}+\boldsymbol{\gamma}}\Delta_{\boldsymbol{\beta}}v(\mathbf{x})\geq 0$.

A.2. Properties on the system state space

i) to v) Trivial

vi) $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_l}(\mathbf{b})$ is equivalent to "for all $\mathbf{x}$ such that $\{\mathbf{x},\mathbf{x}+\mathbf{a}_1,\ldots,\mathbf{x}+\mathbf{a}_l\}\subset\mathcal{X}$, $\mathbf{x}+\mathbf{b}\in\mathcal{X}$". In this assertion we replace $\mathbf{x}$ by $\mathbf{x}+\mathbf{a}_l$ to obtain "for all $\mathbf{x}$ such that $\{\mathbf{x}-\mathbf{a}_l,\mathbf{x}+\mathbf{a}_1-\mathbf{a}_l,\ldots,\mathbf{x}\}\subset\mathcal{X}$, $\mathbf{x}+\mathbf{b}\in\mathcal{X}$'. So $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_l}(\mathbf{b})=\mathtt{R}_{-\mathbf{a}_l,\mathbf{a}_1-\mathbf{a}_l,\ldots,\mathbf{a}_{l-1}-\mathbf{a}_l,\mathbf{0}}(\mathbf{b}-\mathbf{a}_l)$.

Appendix B: Translation operator

With $\mathbf{y}=\mathbf{x}+\mathbf{b}$ and $\forall\mathbf{x},\mathbf{x}+\mathbf{b}\in\mathcal{X}$,

$$\mathcal{T}v(\mathbf{x})=\begin{cases}v(\mathbf{y}+\mathbf{a})+c_a & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ v(\mathbf{y})+c_r & \text{otherwise.}\end{cases}\tag{8}$$

B.1. Propagation of P and N (Cells 1 and 3)

We suppose that v is $\mathtt{P}$ (i.e. $v\geq 0$), then we want find conditions to have $\mathcal{T}$ which propagates $\mathtt{P}$ (i.e. $\mathcal{T}v\geq 0$). Given equation (8), we need to consider two cases:

- if $\mathbf{y}+\mathbf{a}\in\mathcal{X}$, then $\mathcal{T}v\geq 0$ if $c_a\geq 0$
- if $\mathbf{y}+\mathbf{a}\notin\mathcal{X}$, then $\mathcal{T}v\geq 0$ if $c_r\geq 0$. However this case is unreachable if $\mathcal{X}$ is $\mathtt{R}_{-\mathbf{b}}(\mathbf{a})$.

So $\mathcal{T}v\geq 0$ if $|c_a\geq 0|\wedge(\mathtt{R}_{-\mathbf{b}}(\mathbf{a})\vee|c_r\geq 0|)$. In the same way, $\mathcal{T}v\leq 0$ if $|c_a\leq 0|\wedge(\mathtt{R}_{-\mathbf{b}}(\mathbf{a})\vee|c_r\leq 0|)$.

B.2. Propagation of $\mathtt{I}_{\boldsymbol{\epsilon}}$ (Cell 5)

$$\Delta_{\boldsymbol{\epsilon}}\mathcal{T}v(\mathbf{x})=\begin{cases}\Delta_{\boldsymbol{\epsilon}}v(\mathbf{y}+\mathbf{a})+\epsilon_{c_a} & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ \Delta_{\boldsymbol{\epsilon}}v(\mathbf{y})+\epsilon_{c_r} & \text{otherwise}\end{cases}$$

So $\mathcal{T}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ if $|\epsilon_{c_a}\geq 0|\wedge(|\epsilon_{c_r}\geq 0|\vee\mathtt{R}_{-\mathbf{b}}(\mathbf{a}))$

B.3. Propagation of $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ and $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ (Cells 7 and 9)

We make the assumption that $\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}v$ is positive (resp. negative), then we want find conditions to have $\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}\mathcal{T}$ positive (resp. negative).

$$\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}\mathcal{T}v(\mathbf{x})=\begin{cases}\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}v(\mathbf{y}+\mathbf{a}) & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}v(\mathbf{y}) & \text{otherwise}\end{cases}$$

So $\mathcal{T}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ or $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ without condition.

B.4. Propagation of $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ (Cell 11)

We make the assumption that v is $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ (i.e. $\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}} v \geq 0$), then we want find conditions to have $\mathcal{T}$ which propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ (i.e. $\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}\mathcal{T} v \geq 0$).

$$\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}\mathcal{T}v(\mathbf{x}) = \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}} \begin{cases} v(\mathbf{y}+\mathbf{a}) + c_a & \text{if } \mathbf{y}+\mathbf{a} \in \mathcal{X} \\ v(\mathbf{y}) + c_r & \text{otherwise} \end{cases}$$

The four possible cases are described in the following table

	$\mathbf{y}+\mathbf{a} \in \mathcal{X}$	$\mathbf{y}+\mathbf{a} \notin \mathcal{X}$
$\mathbf{y}+\mathbf{a}+\mathbf{d} \in \mathcal{X}$	Case 1	Case 3
$\mathbf{y}+\mathbf{a}+\mathbf{d} \notin \mathcal{X}$	Case 2	Case 4

- Case 1 $= 0$
- Case 2 $= \Delta_{\boldsymbol{\epsilon}}[v(\mathbf{y}+\mathbf{d}) + c_r - v(\mathbf{y}+\mathbf{a}) - c_a]$

$= \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}-\mathbf{a}} v(\mathbf{y}+\mathbf{a}) + \epsilon_{c_r} - \epsilon_{c_a}$

— Positive if $\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0|$

— Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d})$

- Case 3 $= \Delta_{\boldsymbol{\epsilon}}[v(\mathbf{y}+\mathbf{d}+\mathbf{a}) + c_a - v(\mathbf{y}) - c_r]$

$= \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) - \epsilon_{c_r} + \epsilon_{c_a}$

— Positive if $\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0|$

— Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b})$

- Case 4 $= 0$

So $\mathcal{T}$ propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ if

$$(\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0| \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d})) \wedge (\mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0| \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}))$$

B.5. $\mathtt{PM}(\mathcal{T})$ and $\mathtt{NM}(\mathcal{T})$ (Cells 13 and 15)

$$\mathcal{T}v(\mathbf{x}) - v(\mathbf{x}) = \begin{cases} \Delta_{\mathbf{a}+\mathbf{b}} v(\mathbf{x}) + c_a & \text{if } \mathbf{y}+\mathbf{a} \in \mathcal{X} & \text{Case 1} \\ \Delta_{\mathbf{b}} v(\mathbf{x}) + c_r & \text{otherwise} & \text{Case 2} \end{cases}$$

So v is $\mathtt{PM}(\mathcal{T})$ if $|\Delta_{\mathbf{a}+\mathbf{b}} v \geq -c_a| \wedge (|\Delta_{\mathbf{b}} v \geq -c_r| + \mathtt{R}_{-\mathbf{b}}(\mathbf{a}))$ and v is $\mathtt{NM}(\mathcal{T})$ if $[|\Delta_{\mathbf{a}+\mathbf{b}} v \leq -c_a| \wedge (|\Delta_{\mathbf{b}} v \leq -c_r| + \mathtt{R}_{-\mathbf{b}}(\mathbf{a}))$

B.6. $\mathtt{IM}_{\boldsymbol{\epsilon}}(\mathcal{T})$ (Cell 17)

$$\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{T}} v(\mathbf{x}) = \begin{cases} \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{a}+\mathbf{b}} v(\mathbf{x}) + \epsilon_{c_a} & \text{if } \mathbf{y}+\mathbf{a} \in \mathcal{X} \\ \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{b}} v(\mathbf{x}) + \epsilon_{c_r} & \text{otherwise} \end{cases}$$

So, v is $\mathtt{IM}_{\boldsymbol{\epsilon}}(\mathcal{T})$ if $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{a}+\mathbf{b}} \wedge |\epsilon_{c_a} \geq 0| \wedge (\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge |\epsilon_{c_r} \geq 0| \vee \mathtt{R}(\mathbf{a}+\mathbf{b}))$

B.7. $\mathtt{IM}_{\mathbf{d}}(\mathcal{T})$ (Cell 19)

$$\Delta_{\mathbf{d}}\Omega_{\mathcal{T}}v(\mathbf{x}) = \Delta_{\mathbf{d}}\begin{cases}\Delta_{\mathbf{a+b}}v(\mathbf{x}) + c_a & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ \Delta_{\mathbf{b}}v(\mathbf{x}) + c_r & \text{otherwise}\end{cases}$$

The four possible cases are described in the following table

	$\mathbf{y}+\mathbf{a}\in\mathcal{X}$	$\mathbf{y}+\mathbf{a}\notin\mathcal{X}$
$\mathbf{y}+\mathbf{a}+\mathbf{d}\in\mathcal{X}$	Case 1	Case 3
$\mathbf{y}+\mathbf{a}+\mathbf{d}\notin\mathcal{X}$	Case 2	Case 4

- Case 1 $=\Delta_{\mathbf{d}}\Delta_{\mathbf{a+b}}v(\mathbf{x})$
  - Positive if $\mathtt{S}_{\mathbf{d},\mathbf{a+b}}$
- Case 2 $=\Delta_{\mathbf{b}}v(\mathbf{x}+\mathbf{d}) + c_r - \Delta_{\mathbf{b+a}}v(\mathbf{x}) - c_a = \begin{cases}\Delta_{\mathbf{d}}\Delta_{\mathbf{b}}v(\mathbf{x}) - \Delta_{\mathbf{a}}v(\mathbf{x}+\mathbf{b}) + c_r - c_a\\ \Delta_{\mathbf{d-a}}\Delta_{\mathbf{b}}v(\mathbf{x}+\mathbf{a}) - \Delta_{\mathbf{a}}v(\mathbf{x}) + c_r - c_a\end{cases}$
  - Positive if $|\Delta_{\mathbf{a}}v \le c_r - c_a| \wedge (\mathtt{S}_{\mathbf{b},\mathbf{d}} \vee \mathtt{S}_{\mathbf{b},\mathbf{d-a}})$
  - Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a+b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}))$
- Case 3 $=\Delta_{\mathbf{b+a}}v(\mathbf{x}+\mathbf{d}) + c_a - \Delta_{\mathbf{b}}v(\mathbf{x}) - c_r = \begin{cases}\Delta_{\mathbf{d}}\Delta_{\mathbf{b}}v(\mathbf{x}) + \Delta_{\mathbf{a}}v(\mathbf{x}+\mathbf{b}+\mathbf{d}) - c_r + c_a\\ \Delta_{\mathbf{d+a}}\Delta_{\mathbf{b}}v(\mathbf{x}) + \Delta_{\mathbf{a}}v(\mathbf{x}+\mathbf{d}) - c_r + c_a\end{cases}$
  - Positive if $|\Delta_{\mathbf{a}}v \ge c_r - c_a| \wedge (\mathtt{S}_{\mathbf{b},\mathbf{d}} \vee \mathtt{S}_{\mathbf{b},\mathbf{d+a}})$
  - Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a+b+d}}(\mathbf{a}+\mathbf{b}))$
- Case 4 $=\Delta_{\mathbf{d}}\Delta_{\mathbf{b}}v(\mathbf{x})$
  - Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$
  - Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \vee \mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b})$

So, v is $\mathtt{IM}_{\mathbf{d}}(\mathcal{T})$ if

$$\begin{gathered}\mathtt{S}_{\mathbf{d},\mathbf{a+b}} \wedge (\mathtt{S}_{\mathbf{d},\mathbf{b}} \vee \mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \vee \mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}))\\ \wedge(|\Delta_{\mathbf{a}}v \le c_r - c_a| \wedge [\mathtt{S}_{\mathbf{d},\mathbf{b}} \vee \mathtt{S}_{\mathbf{b},\mathbf{d-a}}] \vee \mathtt{R}_{\mathbf{d},\mathbf{a+b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}))\\ \wedge(|\Delta_{\mathbf{a}}v \ge c_r - c_a| \wedge [\mathtt{S}_{\mathbf{d},\mathbf{b}} \vee \mathtt{S}_{\mathbf{b},\mathbf{d+a}}] \vee \mathtt{R}_{\mathbf{d},\mathbf{a+b+d}}(\mathbf{a}+\mathbf{b}))\end{gathered}$$

Appendix C: Choice operator

$$\mathcal{C}v(\mathbf{x}) = \begin{cases}\min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\} & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ v(\mathbf{y}) + c_r, & \text{otherwise}\end{cases} \tag{9}$$

with $\mathbf{y}=\mathbf{x}+\mathbf{b}$ and $\forall\mathbf{x}$, $\mathbf{x}+\mathbf{b}\in\mathcal{X}$. In this section we may use $c_d = c_a - c_b$.

C.1. Propagation of $\mathtt{P}$ and $\mathtt{N}$ (Cells 2 and 4)

We suppose that v poisitve (resp. negative). From equation (9) the condition to have $\mathcal{C}v$ positive (resp. negative) is

$$|c_a \ge 0| \wedge |c_b \ge 0| \wedge (|c_r \ge 0| \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a})) \qquad (\text{resp. } |c_a \le 0| \wedge |c_b \le 0| \wedge (|c_r \le 0| \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a})))$$

C.2. Propagation of $\mathtt{I}_\epsilon$ (Cell 6)

$$\Delta_\epsilon \mathcal{C}v(\mathbf{x}) = \begin{cases} \Delta_\epsilon \min\{v(\mathbf{y})+c_b, v(\mathbf{y}+\mathbf{a})+c_a\} \\ \quad \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X} \\ \Delta_\epsilon v(\mathbf{y})+c_r, \text{ otherwise} \end{cases}$$

The four cases of $\Delta_\epsilon \min\{v(\mathbf{y})+c_b, v(\mathbf{y}+\mathbf{a})+c_a\}$ are described in the following table.

	$\Delta_\mathbf{a} v(\mathbf{y}) \le -c_d$	$\Delta_\mathbf{a} v(\mathbf{y}) \ge -c_d$
$\Delta_\mathbf{a} v'(\mathbf{y}) \le -c_d'$	Case 1	Case 3
$\Delta_\mathbf{a} v'(\mathbf{y}) \ge -c_d'$	Case 2	Case 4

- Case $1 = \Delta_\epsilon v(\mathbf{y}+\mathbf{a})+\epsilon_{c_a}$
  —Positive if $|\epsilon_{c_a} \ge 0|$
- Case $2 = v'(\mathbf{y})+c_b' - v(\mathbf{y}+\mathbf{a}) - c_a \ge \Delta_\epsilon v(\mathbf{y})+\epsilon_{c_b}$
  —Positive if $\left|\epsilon_{c_b} \ge 0\right|$
- Case $3 = v'(\mathbf{y}+\mathbf{a})+c_a' - v(\mathbf{y}) - c_b \ge \Delta_\epsilon v(\mathbf{y}+\mathbf{a})+\epsilon_{c_a}$
  —Positive if $|\epsilon_{c_a} \ge 0|$
- Case 4 $Q = \Delta_\epsilon v(\mathbf{y})+\epsilon_{c_b}$
  —Positive if $\left|\epsilon_{c_b} \ge 0\right|$

Note that when $\Delta_\mathbf{a} v \le -c_d - \epsilon^+_{c_d}$ (resp. $\Delta_\mathbf{a} v \ge -c_d + \epsilon^-_{c_d}$) the cases 2, 3, 4 (resp. 1, 2, 3) are Useless.

So $\mathcal{C}$ propagates $\mathtt{I}_\epsilon$ if

$$\begin{pmatrix} |\epsilon_{c_a} \ge 0| \wedge \left|\epsilon_{c_b} \ge 0\right| \\ \vee \left|\Delta_\mathbf{a} v \le -c_d - \epsilon^+_{c_d}\right| \wedge |\epsilon_{c_a} \ge 0| \\ \vee \left|\Delta_\mathbf{a} v \ge -c_d + \epsilon^-_{c_d}\right| \wedge \left|\epsilon_{c_b} \ge 0\right| \end{pmatrix} \wedge \begin{pmatrix} \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \\ \vee |\epsilon_{c_r} \ge 0| \end{pmatrix}$$

C.3. Propagation of $\mathtt{S}_{\epsilon,-\epsilon}$ and $\mathtt{S}_{\epsilon,\epsilon}$ (Cells 8 and 10)

We make the assumption that $\Delta_\epsilon\Delta_\epsilon v$ is positive (resp. negative) then we want find conditions on v, and ϵ to have $\Delta_\epsilon\Delta_\epsilon\mathcal{C}$ positive (resp. negative).

$$\Delta_\epsilon\Delta_\epsilon \mathcal{C}v(\mathbf{x}) = \begin{cases} \Delta_\epsilon\Delta_\epsilon \min\{v(\mathbf{y})+c_b, v(\mathbf{y}+\mathbf{a})+c_a\} \\ \quad \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X} \\ \Delta_\epsilon\Delta_\epsilon v(\mathbf{y}), \text{ otherwise} \end{cases}$$

We focus on $\Delta_\epsilon\Delta_\epsilon \min\{v(\mathbf{y})+c_b, v(\mathbf{y}+\mathbf{a})+c_a\}$. We use $v''(\mathbf{x})$ (resp. c_b'', c_a'') to denote $v(\mathbf{x}+2\epsilon)$ (resp. $c_b+2\epsilon_{c_b}$, $c_a+2\epsilon_{c_a}$).

$$\begin{aligned}\Delta_\epsilon\Delta_\epsilon \min\{v(\mathbf{y})+c_b, v(\mathbf{y}+\mathbf{a})+c_a\} = {} & \min\{v''(\mathbf{y})+c_b'', v''(\mathbf{y}+\mathbf{a})+c_a''\} \\ & -2\min\{v'(\mathbf{y})+c_b', v'(\mathbf{y}+\mathbf{a})+c_a'\} \\ & +\min\{v(\mathbf{y})+c_b, v(\mathbf{y}+\mathbf{a})+c_a\}\end{aligned}$$

The 8 possible cases are given in the following table.

	$\Delta_{\mathbf{a}} v''(\mathbf{y}) \leq -c_d''$	$\Delta_{\mathbf{a}} v''(\mathbf{y}) \geq -c_d''$
$\Delta_{\mathbf{a}} v'(\mathbf{y}) \leq -c_d'$ $\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c_d$	Case 1	Case 5
$\Delta_{\mathbf{a}} v'(\mathbf{y}) \leq -c_d'$ $\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c_d$	Case 2	Case 6
$\Delta_{\mathbf{a}} v'(\mathbf{y}) \geq -c_d'$ $\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c_d$	Case 3	Case 7
$\Delta_{\mathbf{a}} v'(\mathbf{y}) \geq -c_d'$ $\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c_d$	Case 4	Case 8

- Cases 1 and 8 are positive or negative without condition.
- Case 2 $= v''(\mathbf{y}+\mathbf{a}) + c_a'' - 2(v'(\mathbf{y}+\mathbf{a}) + c_a') + v(\mathbf{y}) + c_b = \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{a}} v(\mathbf{y}) - c_d$
  - —Negative without condition
  - —Useless if $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|$
- Case 3 $= v''(\mathbf{y}+\mathbf{a}) + c_a'' - 2(v'(\mathbf{y}) + c_b') + v(\mathbf{y}+\mathbf{a}) + c_a = \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}+\mathbf{a}) + 2\Delta_{\mathbf{a}} v'(\mathbf{y}) + c_d + 2\epsilon_{c_d}$
  - —Positive without condition
  - —Useless if $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right| \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$
- Case 4 $= v''(\mathbf{y}+\mathbf{a}) + c_a'' - 2(v'(\mathbf{y}) + c_b') + v(\mathbf{y}) + c_b = \Delta_{\mathbf{a}} v''(\mathbf{y}) + \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}) + c_d + 2\epsilon_{c_d}$
  - —Negative without condition
  - —Useless if $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|$
- Case 5 $= v''(\mathbf{y}) + c_b'' - 2(v'(\mathbf{y}+\mathbf{a}) + c_a') + v(\mathbf{y}+\mathbf{a}) + c_a = \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{a}} v''(\mathbf{y}) - c_d - 2\epsilon_{c_d}$
  - —Negative without condition
  - —Useless if $\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$
- Case 6 $= v''(\mathbf{y}) + c_b'' - 2(v'(\mathbf{y}+\mathbf{a}) + c_a') + v(\mathbf{y}) + c_b = \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}) - 2\Delta_{\mathbf{a}} v'(\mathbf{y}+\mathbf{a}) - c_d - 2\epsilon_{c_d}$
  - —Positive without condition
  - —Useless if $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right| \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$
- Case 7 $= v''(\mathbf{y}) + c_b'' - 2(v'(\mathbf{y}) + c_b') + v(\mathbf{y}+\mathbf{a}) + c_a = \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}) + \Delta_{\mathbf{a}} v(\mathbf{y}) + c_d$
  - —Negative without condition
  - —Useless if $\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$

So $\mathcal{C}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ if

$$\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} = 0\right| \vee \left|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^{+}_{c_d}\right| \vee \left|\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^{-}_{c_d}\right|$$

and propagate $\mathtt{S}^{ub}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ if

$$\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right| \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \left|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^{+}_{c_d}\right| \vee \left|\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^{-}_{c_d}\right|$$

C.4. Propagation of $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ (Cell 12)

We make the assumption that v is $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ then we want find conditions on v, and $\boldsymbol{\epsilon}$ to have $\mathcal{C}$ which propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$.

$$\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}\mathcal{C}v(\mathbf{x}) = \Delta_{\mathbf{d}}\begin{cases}\Delta_{\boldsymbol{\epsilon}}\min\{v(\mathbf{y})+c_b, v(\mathbf{y}+\mathbf{a})+c_a\} & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ \Delta_{\boldsymbol{\epsilon}}v(\mathbf{y})+c_r, & \text{otherwise}\end{cases}$$

The 4 possible cases are given in the following table.

	$\mathbf{y}+\mathbf{a}\in\mathcal{X}$	$\mathbf{y}+\mathbf{a}\notin\mathcal{X}$
$\mathbf{y}+\mathbf{a}+\mathbf{d}\in\mathcal{X}$	Case A	Case C
$\mathbf{y}+\mathbf{a}+\mathbf{d}\notin\mathcal{X}$	Case B	Case D

C.4.1. Case A.

$$\begin{aligned}\text{Case A} &= \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}\min\{v(\mathbf{y})+c_b, v(\mathbf{y}+\mathbf{a})+c_a\}\\ &= \min\{v'(\mathbf{y}+\mathbf{d})+c_b{}', v'(\mathbf{y}+\mathbf{a}+\mathbf{d})+c_a{}'\} - \min\{v'(\mathbf{y})+c_b{}', v'(\mathbf{y}+\mathbf{a})+c_a{}'\}\\ &\quad - \min\{v(\mathbf{y}+\mathbf{d})+c_b(, v(\mathbf{y}+\mathbf{a}+\mathbf{d})+c_a\} + \min\{v(\mathbf{y})+c_b{}', v(\mathbf{y}+\mathbf{a})+c_a\}\end{aligned}$$

The 16 possible cases of case A are described in Table 5

Table 5 Possible cases for Case A $= \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}\min\{v(\mathbf{y})+c_b, v(\mathbf{y}+\mathbf{a})+c_a\}$.

Case A	$\Delta_{\mathbf{a}}v'(\mathbf{y}+\mathbf{d})\le -c_d{}'$, $\Delta_{\mathbf{a}}v'(\mathbf{y})\le -c_d{}'$	$\Delta_{\mathbf{a}}v'(\mathbf{y}+\mathbf{d})\le -c_d{}'$, $\Delta_{\mathbf{a}}v'(\mathbf{y})\ge -c_d{}'$	$\Delta_{\mathbf{a}}v'(\mathbf{y}+\mathbf{d})\ge -c_d{}'$, $\Delta_{\mathbf{a}}v'(\mathbf{y})\le -c_d{}'$	$\Delta_{\mathbf{a}}v'(\mathbf{y}+\mathbf{d})\ge -c_d{}'$, $\Delta_{\mathbf{a}}v'(\mathbf{y})\ge -c_d{}'$
$\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})\le -c_d$ $\Delta_{\mathbf{a}}v(\mathbf{y})\le -c_d$	Case 1	Case 5	Case 9	Case 13
$\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})\le -c_d$ $\Delta_{\mathbf{a}}v(\mathbf{y})\ge -c_d$	Case 2	Case 6	Case 10	Case 14
$\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})\ge -c_d$ $\Delta_{\mathbf{a}}v(\mathbf{y})\le -c_d$	Case 3	Case 7	Case 11	Case 15
$\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})\ge -c_d$ $\Delta_{\mathbf{a}}v(\mathbf{y})\ge -c_d$	Case 4	Case 8	Case 12	Case 16

- Case 1 $= \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}v(\mathbf{y}+\mathbf{a})\ge 0$
- Case 2 $= \Delta_{\mathbf{d}}v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d+a}}v(\mathbf{y}) - c_d = \Delta_{\mathbf{d+a}}v'(\mathbf{y}) - \Delta_{\mathbf{d+a}}v(\mathbf{y}) - c_d - \Delta_{\mathbf{a}}v'(\mathbf{y})$
  - —Positive if $\left|\epsilon_{c_d}\ge 0\right| \wedge \mathtt{S}_{\mathbf{d+a},\boldsymbol{\epsilon}}$
  - —Useless if $\mathtt{S}_{\mathbf{a},\mathbf{d}}\vee\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}\wedge\left|\epsilon_{c_d}\ge 0\right|$
- Case 3 $= -\Delta_{\mathbf{d-a}}v(\mathbf{y}+\mathbf{a}) + \Delta_{\mathbf{d}}v'(\mathbf{y}+\mathbf{a}) + c_d = -\Delta_{\mathbf{d}}v(\mathbf{y}+\mathbf{a}) + \Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d}) + \Delta_{\mathbf{d}}v'(\mathbf{y}+\mathbf{a}) + c_d \ge 0$
- Case 4 $= \Delta_{\mathbf{d}}v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}}v(\mathbf{y}) \ge \begin{cases}\Delta_{\mathbf{d}}v(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}}v(\mathbf{y})\\ \Delta_{\mathbf{d}}v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}}v(\mathbf{y}) + \underbrace{\Delta_{\mathbf{a}}v'(\mathbf{y}) - \Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})}_{\ge 0 \text{ if } \epsilon_{c_d}\ge 0}\\ \quad = \Delta_{\mathbf{d+a}}v'(\mathbf{y}) - \Delta_{\mathbf{d+a}}v(\mathbf{y})\end{cases}$

—Positive if $\mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d+a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|$

—Useless if $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|$

- Case 5 $= \Delta_{\mathbf{d+a}} v'(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) + c_d' = \Delta_{\mathbf{d}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) + c_d' + \Delta_{\mathbf{a}} v'(\mathbf{y}) \geq 0$
- Case 6 $= \Delta_{\mathbf{d+a}} v'(\mathbf{y}) - \Delta_{\mathbf{d+a}} v(\mathbf{y}) + c_d' - c_d$

—Positive if $\mathtt{S}_{\mathbf{d+a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|$

—Useless if $\mathtt{S}_{\mathbf{d},\mathbf{a}}$

- Case 7 $= \Delta_{\mathbf{d+a}} v'(\mathbf{y}) - \Delta_{\mathbf{d-a}} v(\mathbf{y}+\mathbf{a}) + c_d + c_d'$

—Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|$

- Case 8 $= -\Delta_{\mathbf{d}} v(\mathbf{y}) + \Delta_{\mathbf{d+a}} v'(\mathbf{y}) + c_d' = -\Delta_{\mathbf{d+a}} v(\mathbf{y}) + \Delta_{\mathbf{d+a}} v'(\mathbf{y}) + \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) + c_d'$

—Positive if $\left|\epsilon_{c_d} \geq 0\right| \wedge \mathtt{S}_{\mathbf{d+a},\boldsymbol{\epsilon}}$

—Useless if $\mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|$

- Case 9 $= \Delta_{\mathbf{d-a}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) - c_d' = \Delta_{\mathbf{d-a}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d-a}} v(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{a}} v(\mathbf{y}) - c_d'$

—Positive if $\mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$

—Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$

- Case 10 $= \Delta_{\mathbf{d-a}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d+a}} v(\mathbf{y}) + c_d' + c_d$

—Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|$

- Case 11 $= -\Delta_{\mathbf{d-a}} v(\mathbf{y}+\mathbf{a}) + \Delta_{\mathbf{d-a}} v'(\mathbf{y}+\mathbf{a}) + c_d - c_d'$

—Positive if $\mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$

—Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}$

- Case 12 $= -\Delta_{\mathbf{d}} v(\mathbf{y}) + \Delta_{\mathbf{d-a}} v'(\mathbf{y}+\mathbf{a}) - c_d'$

$= -\Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{a}} v'(\mathbf{y}) + \Delta_{\mathbf{d}} v'(\mathbf{y}) - c_d \geq 0$

- Case 13 $= \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) \geq \begin{cases} \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) \\ \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) + \underbrace{\Delta_{\mathbf{a}} v'(\mathbf{y}+\mathbf{d}) - \Delta_{\mathbf{a}} v(\mathbf{y})}_{\geq 0 \text{ if } \epsilon_{c_d} \leq 0} \\ \quad = \Delta_{\mathbf{d+a}} v'(\mathbf{y}) - \Delta_{\mathbf{d+a}} v(\mathbf{y}) \end{cases}$

—Positive if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$

—Useless if $\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$

- Case 14 $= \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d+a}} v(\mathbf{y}) + c_d = \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) - c_d \geq 0$
- Case 15 $= \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d-a}} v(\mathbf{y}+\mathbf{a}) + c_d = \Delta_{\mathbf{d-a}} v'(\mathbf{y}) - \Delta_{\mathbf{d-a}} v(\mathbf{y}) + \Delta_{\mathbf{a}} v'(\mathbf{y}) + c_d$

—Positive if $\left|\epsilon_{c_d} \leq 0\right| \wedge \mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}}$

—Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$

- Case 16 $= -\Delta_{\mathbf{d}} v(\mathbf{y}) + \Delta_{\mathbf{d}} v'(\mathbf{y}) \geq 0$

Note that if $\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}$ or $\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}$ there is no condition because only cases 1 and 16 can be reach.

So Case A is positive if

$$
\begin{aligned}
&\left|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^{+}_{c_d}\right| \vee \left|\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^{-}_{c_d}\right| \vee \\
&(\left|\epsilon_{c_d} \geq 0\right| \wedge \mathtt{S}_{\mathbf{d+a},\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|) && \text{(Case 2)} \\
&\wedge(\mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d+a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right| \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|) && \text{(Case 4)} \\
&\wedge(\mathtt{S}_{\mathbf{d+a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right| \vee \mathtt{S}_{\mathbf{d},\mathbf{a}}) && \text{(Case 6)} \\
&\wedge(\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|) && \text{(Case 7)} \\
&\wedge(\left|\epsilon_{c_d} \geq 0\right| \wedge \mathtt{S}_{\mathbf{d+a},\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|) && \text{(Case 8)} \\
&\wedge(\left|\epsilon_{c_d} \leq 0\right| \wedge \mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|) && \text{(Case 9)} \\
&\wedge(\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|) && \text{(Case 10)} \\
&\wedge(\mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}) && \text{(Case 11)} \\
&\wedge(\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|) && \text{(Case 13)} \\
&\wedge(\left|\epsilon_{c_d} \leq 0\right| \wedge \mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|) && \text{(Case 15)}
\end{aligned}
$$

With simplifications this condition reduces to

$$
\begin{gathered}
\left|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^{+}_{c_d}\right| \vee \left|\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^{-}_{c_d}\right| \\
\vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d+a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right| \\
\vee \mathtt{S}_{\mathbf{d+a},\boldsymbol{\epsilon}} \wedge \mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}) \wedge \left|\epsilon_{c_d} = 0\right|
\end{gathered}
$$

C.4.2. Case B.

$$
\begin{aligned}
\text{Case B} &= \Delta_{\boldsymbol{\epsilon}}[\mathcal{C}v(\mathbf{x}+\mathbf{d}) - \mathcal{C}v(\mathbf{x})] \\
&= v'(\mathbf{y}+\mathbf{d}) - v(\mathbf{y}+\mathbf{d}) + \epsilon_{c_r} - \min\{v'(\mathbf{y}) + c_b{}', v'(\mathbf{y}+\mathbf{a}) + c_a{}'\} + \min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\}
\end{aligned}
$$

Case B	$\Delta_{\mathbf{a}} v'(\mathbf{y}) \leq -c_d{}'$,	$\Delta_{\mathbf{a}} v'(\mathbf{y}) \geq -c_d{}'$
$\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c_d$	Case 1	Case 3
$\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c_d$	Case 2	Case 4

- Case 1 $= \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d-a}} v(\mathbf{y}+\mathbf{a}) - \epsilon_{c_a} + \epsilon_{c_r}$
  - —Positive if $\mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0|$
- Case 2 $= \Delta_{\mathbf{d-a}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{y}) - c_a{}' + c_b + \epsilon_{c_r} = \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{a}} v'(\mathbf{y}) - c_a{}' + c_b + \epsilon_{c_r}$
  - —Positive if $\left|\epsilon_{c_r} - \epsilon_{c_b} \geq 0\right|$
  - —Useless if $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right|$
- Case 3 $= \Delta_{\mathbf{d}} v'(\mathbf{y}) - c_b{}' - \Delta_{\mathbf{d-a}} v(\mathbf{y}+\mathbf{a}) + c_a + \epsilon_{c_r} = \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d-a}} v(\mathbf{y}) + \Delta_{\mathbf{a}} v'(\mathbf{y}) + c_a - c_b{}' + \epsilon_{c_r}$
  - —Positive if $\mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0|$
  - —Useless if $\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right|$
- Case 4 $= \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d}} v(\mathbf{x}) - \epsilon_{c_b} + \epsilon_{c_r}$
  - —Positive if $\left|\epsilon_{c_r} - \epsilon_{c_b} \geq 0\right|$

Note that when $\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^{+}_{c_d}$ (resp. $\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^{-}_{c_d}$) the cases 2, 3, 4 (resp. 1, 2, 3) are Useless. So case B is

- Positive if

$$
\begin{gathered}
\mathtt{S}_{\mathbf{d-a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0| \wedge \left|\epsilon_{c_r} - \epsilon_{c_b} \geq 0\right| \\
\vee \left|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^{+}_{c_d}\right| \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d-a}} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0| \\
\vee \left|\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^{-}_{c_d}\right| \wedge \left|\epsilon_{c_r} - \epsilon_{c_b} \geq 0\right|
\end{gathered}
$$

- Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a+b}}(\mathbf{a}+\mathbf{b}+\mathbf{d})$

C.4.3. Case C.

$$\begin{aligned}\text{Case C} &= \Delta_{\boldsymbol{\epsilon}}[\mathcal{C}v(\mathbf{x}+\mathbf{d}) - \mathcal{C}v(\mathbf{x})] \\ &= \Delta_{\boldsymbol{\epsilon}}[\mathcal{C}v(\mathbf{x}+\mathbf{d}) - v(\mathbf{y})] - \epsilon_{c_r} \\ &= \min\{v'(\mathbf{y}+\mathbf{d}) + c_b{}', v'(\mathbf{y}+\mathbf{d}+\mathbf{a}) + c_a{}'\} \\ &\quad - \min\{v(\mathbf{y}+\mathbf{d}) + c_b, v(\mathbf{y}+\mathbf{d}+\mathbf{a}) + c_a\} \\ &\quad - v'(\mathbf{y}) + v(\mathbf{y}) - \epsilon_{c_r}\end{aligned}$$

Case C	$\Delta_{\mathbf{a}} v'(\mathbf{y}+\mathbf{d}) \leq -c'$,	$\Delta_{\mathbf{a}} v'(\mathbf{y}+\mathbf{d}) \geq -c'$
$\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \leq -c$	Case 1	Case 3
$\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \geq -c$	Case 2	Case 4

- Case 1 $= \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) + \epsilon_{c_a} - \epsilon_{c_r}$

  —Positive if $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}+\mathbf{a}} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0|$
- Case 2 $= \Delta_{\mathbf{d}+\mathbf{a}} v'(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}) + c_a{}' - c_b - \epsilon_{c_r} = \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) + \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) + c_a{}' - c_b - \epsilon_{c_r}$

  —Positive if $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}+\mathbf{a}} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0|$

  —Useless if $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{a}} \wedge \left|\epsilon_{c_d} \geq 0\right|$
- Case 3 $\Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) - c_a + c_b{}' - \epsilon_{c_r} = \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) - c_a + c_b{}' - \epsilon_{c_r}$

  —Positive if $\left|\epsilon_{c_b} - \epsilon_{c_r} \geq 0\right|$

  —Useless if $\mathtt{S}^{ub}_{\boldsymbol{\epsilon},\mathbf{a}} \wedge \left|\epsilon_{c_d} \leq 0\right|$
- Case 4 $= \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d}} v(\mathbf{y}) + \epsilon_{c_b} - \epsilon_{c_r}$

  —Positive if $\left|\epsilon_{c_b} - \epsilon_{c_r} \geq 0\right|$

Note that when $\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}$ (resp.$\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}$) the cases 2, 3, and 4 (resp. 1, 2, and 3) are Useless. So case C is

- Positive if

$$\begin{gathered}\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}+\mathbf{a}} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0| \wedge \left|\epsilon_{c_b} - \epsilon_{c_r} \geq 0\right| \\ \vee \left|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}\right| \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}+\mathbf{a}} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0| \\ \vee \left|\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}\right| \wedge \left|\epsilon_{c_b} - \epsilon_{c_r} \geq 0\right|\end{gathered}$$

- Useless if $\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b})$

C.4.4. Case D.

$$\text{Case D} = \Delta_{\boldsymbol{\epsilon}}[\mathcal{C}v(\mathbf{x}+\mathbf{d}) - \mathcal{C}v(\mathbf{x})] = \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{d}} v(\mathbf{x}) \geq 0$$

C.4.5. Conclusion. The operator $\mathcal{C}$ propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ if,

$$\begin{gathered}\begin{pmatrix}\left|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}\right| \vee \left|\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}\right| \\ \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right| \\ \vee \mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}} \wedge \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}) \wedge \left|\epsilon_{c_d} = 0\right|\end{pmatrix} \\ \wedge \begin{pmatrix}\mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0| \wedge \left|\epsilon_{c_r} - \epsilon_{c_b} \geq 0\right| \\ \vee \left|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}\right| \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}-\mathbf{a}} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0| \\ \vee \left|\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}\right| \wedge \left|\epsilon_{c_r} - \epsilon_{c_b} \geq 0\right| \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d})\end{pmatrix} \wedge \begin{pmatrix}\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}+\mathbf{a}} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0| \wedge \left|\epsilon_{c_b} - \epsilon_{c_r} \geq 0\right| \\ \vee \left|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}\right| \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}+\mathbf{a}} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0| \\ \vee \left|\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}\right| \wedge \left|\epsilon_{c_b} - \epsilon_{c_r} \geq 0\right| \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b})\end{pmatrix}\end{gathered}$$

We can simplify this results because if $|\Delta_{\mathbf{a}} \leq -c_d|$ the state $\mathbf{x}+\mathbf{a}+\mathbf{b}$ is always chosen in the minimization, so the operator is equivalent to $\mathcal{T}$ (plus the cost c_a), and if $|\Delta_{\mathbf{a}} \leq -c_d|$ the state $\mathbf{x}+\mathbf{a}+\mathbf{b}$ is never chosen in the minimization, so the operator is equivalent to $\mathcal{T}$ or $\mathcal{C}$ with $\mathbf{a}=\mathbf{0}$. So we can consider that $|\Delta_{\mathbf{a}} \leq -c_d| = |\Delta_{\mathbf{a}} \geq -c_d| = false$. Then the relation reduces to

$$\begin{pmatrix} \mathtt{S}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \leq 0\right| \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}} \wedge \left|\epsilon_{c_d} \geq 0\right| \\ \vee \mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}} \wedge \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}) \wedge \left|\epsilon_{c_d} = 0\right| \end{pmatrix}$$
$$\wedge \begin{pmatrix} \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0| \wedge \left|\epsilon_{c_r} - \epsilon_{c_b} \geq 0\right| \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \end{pmatrix} \wedge \begin{pmatrix} \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}+\mathbf{a}} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0| \wedge \left|\epsilon_{c_b} - \epsilon_{c_r} \geq 0\right| \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$$

C.5. $\mathtt{PM}(\mathcal{T})$ and $\mathtt{NM}(\mathcal{T})$ (Cells 14 and 16)

$$\mathcal{C}v(\mathbf{x}) - v(\mathbf{x}) = \begin{cases} \min\{\Delta_{\mathbf{b}} v(\mathbf{x}) + c_b, \Delta_{\mathbf{a}+\mathbf{b}} v(\mathbf{x}) + c_a\} & \text{if } \mathbf{x}+\mathbf{a}+\mathbf{b} \in \mathcal{X} \\ \Delta_{\mathbf{b}} v(\mathbf{x}) + c_r, & \text{otherwise} \end{cases}$$

So v is $\mathtt{PM}(\mathcal{C})$ if

$$(|\Delta_{\mathbf{b}} v \geq -c_b| \vee |\Delta_{\mathbf{a}} v \leq -c_d|) \wedge (|\Delta_{\mathbf{a}+\mathbf{b}} v \geq -c_a| \vee |\Delta_{\mathbf{a}} v \geq -c_d|) \wedge (|\Delta_{\mathbf{b}} v \geq -c_r| \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}))$$

and v is $\mathtt{NM}(\mathcal{C})$ if

$$\left(|\Delta_{\mathbf{b}} v \leq -c_b| \wedge \overline{|\Delta_{\mathbf{a}} v \leq -c_d|} \vee |\Delta_{\mathbf{a}+\mathbf{b}} v \leq -c_a| \wedge \overline{|\Delta_{\mathbf{a}} v \geq -c_d|}\right) \wedge (|\Delta_{\mathbf{b}} v \leq -c_r| \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}))$$

C.6. $\mathtt{IM}_{\boldsymbol{\epsilon}}(\mathcal{T})$ (Cell 18)

$$\Delta_{\boldsymbol{\epsilon}} \Omega_{\mathcal{C}} v(\mathbf{x}) = \begin{cases} \Delta_{\boldsymbol{\epsilon}} \min\{\Delta_{\mathbf{a}+\mathbf{b}} v(\mathbf{x}) + c_a, \Delta_{\mathbf{b}} v(\mathbf{x}) + c_b\} & \text{if } \mathbf{y}+\mathbf{a} \in \mathcal{X} \\ \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{b}} v(\mathbf{x}) + \epsilon_{c_r} & \text{otherwise} \end{cases}$$

The 4 possible cases for $\Delta_{\boldsymbol{\epsilon}} \min\{\Delta_{\mathbf{a}+\mathbf{b}} v(\mathbf{x}) + c_a, \Delta_{\mathbf{b}} v(\mathbf{x}) + c_b\}$ are given in the following table.

	$\Delta_{\mathbf{a}} v'(\mathbf{y}) \leq -c_d$	$\Delta_{\mathbf{a}} v'(\mathbf{y}) \geq -c_d$
$\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c_d$	Case 1	Case 3
$\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c_d$	Case 2	Case 4

- Case 1 $= \Delta_{\boldsymbol{\epsilon}} \Delta_{\mathbf{a}+\mathbf{b}} v(\mathbf{x}) + \epsilon_{c_a}$
  - Positive if $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}+\mathbf{a}} \wedge |\epsilon_{c_a} \geq 0|$
- Case 2 $= \Delta_{\mathbf{a}+\mathbf{b}} v'(\mathbf{x}) + c_a{}' - \Delta_{\mathbf{b}} v(\mathbf{x}) + c_b$
  - Useless if $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{a}}$
- Case 3 $= \Delta_{\mathbf{b}} v'(\mathbf{x}) + c_b{}' - \Delta_{\mathbf{a}+\mathbf{b}} v(\mathbf{x}) - c_a \geq \Delta_{\boldsymbol{\epsilon}} v(\mathbf{x}+\mathbf{b}) - \Delta_{\boldsymbol{\epsilon}} v(\mathbf{x}) + \epsilon_{c_b}$
  - Positive if $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge \left|\epsilon_{c_b} \geq 0\right|$
  - Useless if $\mathtt{S}^{ub}_{\boldsymbol{\epsilon},\mathbf{a}}$

- Case 4 $= \Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{b}} v(\mathbf{x}) + \epsilon_{c_b}$

  —Positive if $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge \left|\epsilon_{c_b} \geq 0\right|$

Note that when $\Delta_{\mathbf{a}} v \leq -c_d$ (resp.$\Delta_{\mathbf{a}} v \geq -c_d$) the cases 2, 3, and 4 (resp. 1, 2, and 3) are Useless. So $\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{C}} v$ is positive if

$$\begin{pmatrix} \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{a}} \wedge |\epsilon_{c_a} \geq 0| \wedge \left|\epsilon_{c_b} \geq 0\right| \\ \vee |\Delta_{\mathbf{a}} v \leq -c_d| \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}+\mathbf{a}} \wedge |\epsilon_{c_a} \geq 0| \\ \vee |\Delta_{\mathbf{a}} v \geq -c_d| \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge \left|\epsilon_{c_b} \geq 0\right| \end{pmatrix} \wedge \begin{pmatrix} \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge |\epsilon_{c_r} \geq 0| \\ \vee \mathtt{R}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$$

C.7. $\mathtt{IM}_{\mathbf{d}}(\mathcal{T})$ (Cell 20)

$$\Delta_{\mathbf{d}}\Omega_{\mathcal{C}} v(\mathbf{x}) = \Delta_{\mathbf{d}}(\mathcal{C}v(\mathbf{x}) - v(\mathbf{x}))$$

The 4 possible cases are given in the following table.

	$\mathbf{y}+\mathbf{a} \in \mathcal{X}$	$\mathbf{y}+\mathbf{a} \notin \mathcal{X}$
$\mathbf{y}+\mathbf{a}+\mathbf{d} \in \mathcal{X}$	Case A	Case C
$\mathbf{y}+\mathbf{a}+\mathbf{d} \notin \mathcal{X}$	Case B	Case D

C.7.1. Case A.

$$\begin{aligned} \Delta_{\mathbf{d}}\Omega_{\mathcal{C}} v(\mathbf{x}) = \min\{v(\mathbf{y}+\mathbf{d}) + c_b, v(\mathbf{y}+\mathbf{d}+\mathbf{a}) + c_a\} - v(\mathbf{x}+\mathbf{d}) \\ - \min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\} + v(\mathbf{x}) \end{aligned}$$

The 4 possible cases are given in the following table.

	$\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c_d$	$\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c_d$
$\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \leq -c_d$	Case 1	Case 3
$\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \geq -c_d$	Case 2	Case 4

- Case 1 $= \Delta_{\mathbf{d}} v(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{x}) = \Delta_{\mathbf{d}} v(\mathbf{x}+\mathbf{b}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{x})$

  —Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}+\mathbf{a}}$

- Case 2 $= \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{x}) - \Delta_{\mathbf{a}} v(\mathbf{y}) - c_d \geq \Delta_{\mathbf{d}} v(\mathbf{x}+\mathbf{b}) - \Delta_{\mathbf{d}} v(\mathbf{x})$

  —Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$

  —Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}$

- Case 3 $= \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{x}) + \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) + c_d \leq \Delta_{\mathbf{d}} v(\mathbf{x}+\mathbf{b}) - \Delta_{\mathbf{d}} v(\mathbf{x})$

  —Useless if $\mathtt{S}_{\mathbf{d},\mathbf{a}}$

- Case 4 $= \Delta_{\mathbf{d}} v(\mathbf{x}+\mathbf{b}) - \Delta_{\mathbf{d}} v(\mathbf{x})$

  —Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$

Note that when $\Delta_{\mathbf{a}} v \leq -c_d$ (resp.$\Delta_{\mathbf{a}} v \geq -c_d$) the cases 2, 3, and 4 (resp. 1, 2, and 3) are Useless. So Case A is

- Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee |\Delta_{\mathbf{a}} v \leq -c_d| \wedge \mathtt{S}_{\mathbf{d},\mathbf{b}+\mathbf{a}} \vee |\Delta_{\mathbf{a}} v \geq -c_d| \wedge \mathtt{S}_{\mathbf{d},\mathbf{b}}$

C.7.2. Case B. Case B$= v(\mathbf{y}+\mathbf{d}) + c_r - v(\mathbf{x}+\mathbf{d}) - \mathcal{C}v(\mathbf{x}) + v(\mathbf{x})$

- If $\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c_d$ then Case B $= \begin{cases} \Delta_{\mathbf{d}}\Delta_{\mathbf{b}} v(\mathbf{x}) - \Delta_{\mathbf{a}} v(\mathbf{y}) + c_r - c_a \geq \Delta_{\mathbf{b}}\Delta_{\mathbf{d}} v(\mathbf{x}) + c_r + c_b \\ \Delta_{\mathbf{d}-\mathbf{a}}\Delta_{\mathbf{b}} v(\mathbf{x}+\mathbf{a}) - \Delta_{\mathbf{a}} v(\mathbf{x}) + c_r - c_a \end{cases}$

  —Positive if $\big(\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_r + c_b \geq 0| \vee \mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}} \wedge |\Delta_{\mathbf{a}} v \leq c_r - c_a|\big)$

  —Useless if $\Delta_{\mathbf{a}} v \geq -c_d$

- If $\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c_d$ then Case B $= \Delta_{\mathbf{b}}\Delta_{\mathbf{d}} v(\mathbf{x}) + c_r - c_b$

  —Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_r - c_b \geq 0|$

  —Useless if $\Delta_{\mathbf{a}} v \leq -c_d$

So Case B is

- Positive if

$$\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_r + c_b \geq 0| \vee \mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}} \wedge |\Delta_{\mathbf{a}} v \leq c_r - c_a| \vee |\Delta_{\mathbf{a}} v \geq -c_d| \wedge (\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_r - c_b \geq 0| \vee |\Delta_{\mathbf{a}} v \leq -c_d|)$$

- Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d})$

C.7.3. Case C. Case C $= \mathcal{C}v(\mathbf{x}+\mathbf{d}) - v(\mathbf{x}+\mathbf{d}) - v(\mathbf{y}) - c_r + v(\mathbf{x})$

- If $\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \leq -c_d$ then Case C $= \begin{cases} \Delta_{\mathbf{d}}\Delta_{\mathbf{b}} v(\mathbf{x}) + \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) - c_r + c_a \\ \Delta_{\mathbf{d}+\mathbf{a}}\Delta_{\mathbf{b}} v(\mathbf{x}) + \Delta_{\mathbf{a}} v(\mathbf{x}+\mathbf{d}) - c_r + c_a \end{cases}$

  —Positive if $|\Delta_{\mathbf{a}} v \geq c_r - c_a| \wedge (\mathtt{S}_{\mathbf{b},\mathbf{d}} \vee \mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}})$

  —Useless if $\Delta_{\mathbf{a}} v \geq -c_d$

- If $\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \geq -c_d$ then Case C $= \Delta_{\mathbf{b}}\Delta_{\mathbf{d}} v(\mathbf{x}) - c_r + c_b$

  —Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_b - c_r \geq 0|$

  —Useless if $\Delta_{\mathbf{a}} v \leq -c_d$

So Case C is

- Positive if $(|\Delta_{\mathbf{a}} v \geq c_r - c_a| \wedge (\mathtt{S}_{\mathbf{b},\mathbf{d}} \vee \mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}}) \vee |\Delta_{\mathbf{a}} v \geq -c_d|) \wedge (\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_b - c_r \geq 0| \vee |\Delta_{\mathbf{a}} v \leq -c_d|)$
- Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b})$

C.7.4. Case D. Case D $= \Delta_{\mathbf{d}}\Delta_{\mathbf{b}} v(\mathbf{x})$

- Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$
- Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b})) \vee \mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}+\mathbf{d}))$

C.7.5. Conclusion. $\Delta_{\mathbf{d}}\Omega_{\mathcal{C}} v \geq 0$ if,

$$\begin{pmatrix} \mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge \mathtt{S}_{\mathbf{d},\mathbf{a}} \\ \vee |\Delta_{\mathbf{a}} v \leq -c_d| \wedge \mathtt{S}_{\mathbf{d},\mathbf{b}+\mathbf{a}} \\ \vee |\Delta_{\mathbf{a}} v \geq -c_d| \wedge \mathtt{S}_{\mathbf{d},\mathbf{b}} \end{pmatrix} \wedge (\mathtt{S}_{\mathbf{d},\mathbf{b}} \vee \mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \vee \mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}))$$
$$\wedge \begin{pmatrix} \begin{pmatrix} \mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_r + c_b \geq 0| \\ \vee \mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}} \wedge |\Delta_{\mathbf{a}} v \leq c_r - c_a| \\ \vee |\Delta_{\mathbf{a}} v \geq -c_d| \end{pmatrix} \\ \wedge (\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_r - c_b \geq 0| \vee |\Delta_{\mathbf{a}} v \leq -c_d|) \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \end{pmatrix} \wedge \begin{pmatrix} \begin{pmatrix} \mathtt{S}_{\mathbf{b},\mathbf{d}} \wedge |\Delta_{\mathbf{a}} v \geq c_r - c_a| \\ \vee \mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}} \wedge |\Delta_{\mathbf{a}} v \geq c_r - c_a|) \\ \vee |\Delta_{\mathbf{a}} v \geq -c_d| \end{pmatrix} \\ \wedge (\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_b - c_r \geq 0| \vee |\Delta_{\mathbf{a}} v \leq -c_d|) \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$$

With $|\Delta_{\mathbf{a}} v \leq -c_d| = |\Delta_{\mathbf{a}} v \geq -c_d| = false$ this expression reduces to

$$(|c_r \geq 0| \wedge |c_b = 0| \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d})) \wedge \mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge \mathtt{S}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b})$$

Appendix D: Admission control

$$\begin{aligned}
\boldsymbol{\mathcal{M}}v &= \mathcal{H} + \mu\mathcal{O}_0 v + \textstyle\sum_{i=1}^{n} \lambda_i \mathcal{O}_i v + p_0 v,\\
\mathcal{H}(\mathbf{x}) &= hx,\\
\mathcal{O}_0 v(\mathbf{x}) &= \mathcal{T}v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a} = -\mathbf{e}_1, \mathbf{b} = \mathbf{0},\\ c_a = c_r = 0, \end{cases}\\
\mathcal{O}_i v(\mathbf{x}) &= \mathcal{C}v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a} = \mathbf{e}_1, \mathbf{b} = \mathbf{0},\\ c_b = c_i, c_a = c_r = 0. \end{cases}
\end{aligned}$$

The state space is $\mathcal{S}_1 = \mathbb{Z}^+$.

From Stidham (1985) we know that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}$ and $\mathtt{I}_{\mathbf{e}_1}$.

D.1. Proof of Theorem 1

D.1.1. Monotonicity. We look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{M}}$ that propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$. From Proposition 2 we obtain that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ if the following condition is satisfied, knowing that v is $\mathtt{I}_{\boldsymbol{\epsilon}}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}$, and $\mathtt{I}_{\mathbf{e}_1}$.

$$\begin{aligned}
&|\Delta_{\boldsymbol{\epsilon}}(hx) \geq 0|\\
&\bigwedge \left[\begin{array}{c} |\mathcal{O}_0 \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \bigwedge \begin{pmatrix} |\epsilon_\mu < 0| \wedge |\Omega_{\mathcal{O}_0} v \leq 0| \\ \vee |\epsilon_\mu > 0| \wedge |\Omega_{\mathcal{O}_0} v \geq 0| \\ \vee |\epsilon_\mu = 0| \end{pmatrix} \end{array} \right] \bigwedge_{i=1}^{l} \left[\begin{array}{c} |\mathcal{O}_i \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \bigwedge \begin{pmatrix} |\epsilon_{\lambda_i} < 0| \wedge |\Omega_{\mathcal{O}_i} v \leq 0| \\ \vee |\epsilon_{\lambda_i} > 0| \wedge |\Omega_{\mathcal{O}_i} v \geq 0| \\ \vee |\epsilon_{\lambda_i} = 0| \end{pmatrix} \end{array} \right] \bigwedge \begin{pmatrix} |\epsilon_\eta < 0| \wedge |v \text{ is } \mathtt{P}| \\ \vee |\epsilon_\eta > 0| \wedge |v \text{ is } \mathtt{N}| \\ \vee |\epsilon_\eta = 0| \end{pmatrix}.
\end{aligned} \tag{10}$$

From Table 4 we obtain the following relations.

- $|\Delta_{\boldsymbol{\epsilon}}(hx) \geq 0| = |\epsilon_h \geq 0|$
- $|\mathcal{O}_0$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}| = true$ (see cell 5).
- $|\Omega_{\mathcal{O}_0} v \leq 0| = |\Delta_{-\mathbf{e}_1} v \leq 0| = true$ (see cell 15).
- $|\Omega_{\mathcal{O}_0} v \geq 0| = |\Delta_{-\mathbf{e}_1} v \geq 0| = false$ (see cell 13).
- $|\mathcal{O}_i$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}| = |\epsilon_{c_i} \geq 0|$ because $\mathtt{R}(\mathbf{e}_1) = true$ (see cell 6).
- $|\Omega_{\mathcal{O}_i} v \leq 0| = |\Delta_{\mathbf{e}_1} v \leq 0| = false$ (see cell 16).
- $|\Omega_{\mathcal{O}_i} v \geq 0| = |\Delta_{\mathbf{e}_1} v \geq 0| = true$ (see cell 14).
- $|v$ is $\mathtt{P}| = true$ because costs are positive (see cells 1 and 2).
- $|v$ is $\mathtt{N}| = false$ because costs are not negative (see cells 3 and 4).

So equation (10) can be reduced to

$$|\epsilon_h \geq 0| \wedge |\epsilon_\mu \leq 0| \wedge |\epsilon_\eta \leq 0| \bigwedge_{i=1}^{l} (|\epsilon_{c_i} \geq 0| \wedge |\epsilon_{\lambda_i} \geq 0|) \tag{11}$$

Conclusion, the optimal value function is increasing in the arrival rates λ_i, the rejection costs c_i, the holding cost h and decreasing in the service rate μ and the discount rate η.

D.1.2. Convexity/Concavity. First we look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{M}}$ that propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$. However $|\mathcal{O}_i \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}| = false$, so $\boldsymbol{\mathcal{M}}$ does not propagate $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ (see Proposition 3 and cell 10 in Table 4).

Now we look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{M}}$ that propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$. From Proposition 3 we obtain that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ if the following condition is satisfied, knowing that v is $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}$, and $\mathtt{I}_{\mathbf{e}_1}$.

$$|\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}(hx) \le 0| \bigwedge \left[\begin{array}{c} |\mathcal{O}_0 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| \\ \bigwedge \begin{pmatrix} |\epsilon_\mu > 0| \wedge |\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_0} v \le 0| \\ \vee |\epsilon_\mu < 0| \wedge |\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_0} v \ge 0| \\ \vee |\epsilon_\mu = 0| \end{pmatrix} \end{array}\right] \bigwedge_{i=1}^{l} \left[\begin{array}{c} |\mathcal{O}_i \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| \\ \bigwedge \begin{pmatrix} |\epsilon_{\lambda_i} > 0| \wedge |\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_i} v \le 0| \\ \vee |\epsilon_{\lambda_i} < 0| \wedge |\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_i} v \ge 0| \\ \vee |\epsilon_{\lambda_i} = 0| \end{pmatrix} \end{array}\right] \bigwedge \begin{pmatrix} |\epsilon_\eta > 0| \wedge |v \text{ is } \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \vee |\epsilon_\eta < 0| \wedge |v \text{ is } \mathtt{I}_{-\boldsymbol{\epsilon}}| \\ \vee |\epsilon_\eta = 0| \end{pmatrix}. \quad (12)$$

From Table 4 we obtain the following relations.

- $|\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}(hx) \le 0| = true$.
- $|\mathcal{O}_0 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| = true$ (see cell 9).
- $|\Delta_{-\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_0} v \ge 0| = \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$ (see cell 17).
- $|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_0} v \ge 0| = \mathtt{S}_{-\boldsymbol{\epsilon},\mathbf{e}_1}$ (see cell 17).
- $|\mathcal{O}_i \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| = \mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}} \wedge |\epsilon_{c_i} \le 0| \vee \mathtt{S}^{ub}_{\mathbf{e}_1,\boldsymbol{\epsilon}} \wedge |\epsilon_{c_i} \ge 0|$ (see cell 10).
- $|\Delta_{-\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_i} v \ge 0| = \mathtt{S}_{-\boldsymbol{\epsilon},\mathbf{e}_1} \wedge |\epsilon_{c_i} \le 0|$ (see cell 18).
- $|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_i} v \ge 0| = \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1} \wedge |\epsilon_{c_i} \ge 0|$ (see cell 18).
- $|v \text{ is } \mathtt{I}_{\boldsymbol{\epsilon}}|$ if (see equation 11) $|\epsilon_h \ge 0| \wedge |\epsilon_\mu \le 0| \wedge |\epsilon_\eta \le 0| \bigwedge_{i=1}^{l} |\epsilon_{c_i} \ge 0| \wedge |\epsilon_{\lambda_i} \ge 0|$.
- $|v \text{ is } \mathtt{I}_{-\boldsymbol{\epsilon}}|$ if (see equation 11) $|\epsilon_h \le 0| \wedge |\epsilon_\mu \ge 0| \wedge |\epsilon_\eta \ge 0| \bigwedge_{i=1}^{l} |\epsilon_{c_i} \le 0| \wedge |\epsilon_{\lambda_i} \le 0|$.

So equation (12) reduces to

$$\begin{pmatrix} |\epsilon_\mu > 0| \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1} \\ \vee |\epsilon_\mu < 0| \wedge \mathtt{S}_{-\boldsymbol{\epsilon},\mathbf{e}_1} \\ \vee |\epsilon_\mu = 0| \end{pmatrix} \bigwedge_{i=1}^{l} \left[\begin{array}{c} \mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}} \wedge |\epsilon_{c_i} \le 0| \vee \mathtt{S}^{ub}_{\mathbf{e}_1,\boldsymbol{\epsilon}} \wedge |\epsilon_{c_i} \ge 0| \\ \bigwedge \begin{pmatrix} |\epsilon_{\lambda_i} > 0| \wedge \mathtt{S}_{-\boldsymbol{\epsilon},\mathbf{e}_1} \wedge |\epsilon_{c_i} \le 0| \\ \vee |\epsilon_{\lambda_i} < 0| \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1} \wedge |\epsilon_{c_i} \ge 0| \\ \vee |\epsilon_{\lambda_i} = 0| \end{pmatrix} \end{array}\right] \bigwedge |\epsilon_\eta = 0| \ . \quad (13)$$

In the following section (see equation 15) we will see that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$ if

$$|\epsilon_h \ge 0| \wedge |\epsilon_{c_i} \ge 0| \wedge |\epsilon_{\lambda_i} \ge 0| \wedge |\epsilon_\mu \le 0| \wedge |\epsilon_\eta \le 0|,$$

so equation (13) reduces to

$$|\epsilon_{c_i} = 0| \wedge |\epsilon_{\lambda_i} = 0| \wedge |\epsilon_\mu = 0| \wedge |\epsilon_\eta = 0| \ .$$

Conclusion, the optimal value function is concave in the holding cost h.

D.1.3. Monotonicity of the optimal policy. We look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{M}}$ that propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$. From Proposition 3 we obtain that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$ if the following condition is satisfied, knowing that v is $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}$, and $\mathtt{I}_{\mathbf{e}_1}$.

$$\begin{aligned}&|\Delta_{\mathbf{e}_1}\Delta_{\boldsymbol{\epsilon}}(hx)\geq 0|\\ &\bigwedge\left[\begin{matrix}|\mathcal{O}_0 \text{ propagates } \mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}|\\ \bigwedge\begin{pmatrix}|\epsilon_\mu<0|\wedge|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_0}v\leq 0|\\ \vee|\epsilon_\mu>0|\wedge|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_0}v\geq 0|\\ \vee|\epsilon_\mu=0|\end{pmatrix}\end{matrix}\right]\bigwedge_{i=1}^{l}\left[\begin{matrix}|\mathcal{O}_i \text{ propagates } \mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}|\\ \bigwedge\begin{pmatrix}|\epsilon_{\lambda_i}<0|\wedge|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_i}v\leq 0|\\ \vee|\epsilon_{\lambda_i}>0|\wedge|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_i}v\geq 0|\\ \vee|\epsilon_{\lambda_i}=0|\end{pmatrix}\end{matrix}\right]\bigwedge\begin{pmatrix}|\epsilon_\eta<0|\wedge|v \text{ is } \mathtt{I}_{\mathbf{e}_1}|\\ \vee|\epsilon_\eta>0|\wedge|v \text{ is } \mathtt{I}_{-\mathbf{e}_1}|\\ \vee|\epsilon_\eta=0|\end{pmatrix}.\end{aligned} \tag{14}$$

From Table 4 we obtain the following relations.

- $|\Delta_{\mathbf{e}_1}\Delta_{\boldsymbol{\epsilon}}(hx)\geq 0|=|\epsilon_h\geq 0|$
- $|\mathcal{O}_0$ propagates $\mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}|=true$ (see cell 11).
- $|\Delta_{-\mathbf{e}_1}\Omega_{\mathcal{O}_0}v\leq 0|=\mathtt{S}_{-\mathbf{e}_1,-\mathbf{e}_1}\wedge|\Delta_{-\mathbf{e}_1}v\leq 0|=true$ (see cell 19).
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_0}v\geq 0|=\mathtt{S}_{\mathbf{e}_1,-\mathbf{e}_1}\wedge\cdots=false$ (see cell 19).
- $|\mathcal{O}_i$ propagates $\mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}|=\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}\wedge|\epsilon_{c_i}\geq 0|$ (see cell 12).
- $|\Delta_{-\mathbf{e}_1}\Omega_{\mathcal{O}_i}v\geq 0|=\mathtt{S}_{-\mathbf{e}_1,\mathbf{e}_1}=false$ (see cell 20).
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_i}v\geq 0|=\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}=true$ (see cell 20).
- $|v$ is $\mathtt{I}_{\mathbf{e}_1}|=true$ (Stidham 1985).
- $|v$ is $\mathtt{I}_{-\mathbf{e}_1}|=false$ (Stidham 1985).

So equation (14) can be reduced, and $\boldsymbol{\mathcal{M}}$ propages $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$ if

$$|\epsilon_h\geq 0|\wedge|\epsilon_{c_i}\geq 0|\wedge|\epsilon_{\lambda_i}\geq 0|\wedge|\epsilon_\mu\leq 0|\wedge|\epsilon_\eta\leq 0|. \tag{15}$$

Given that the optimal thresholds t_i decrease if

$$|\boldsymbol{\mathcal{M}} \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}}|\wedge|\epsilon_{c_i}\leq 0|,$$

the optimal thresholds t_i are decreasing in the arrival rate λ_i, the holding cost h, and increasing in the service rate μ and the discount rate η.

D.2. Proof of Theorem 2

D.2.1. Effect of λ and μ : Piecewise convexity. Let $[\mu_l,\mu_u]$ (resp. $[\lambda_l,\lambda_u]$) be a set such that for all $\mu\in[\mu_l,\mu_u]$ (resp. $\lambda_i\in[\lambda_l,\lambda_u]$) the optimal thresholds S_i^* do not change. For all $\mu\in[\mu_l,\mu_u]$ (resp. $\lambda_i\in[\lambda_l,\lambda_u]$) the MDP formulation can be rewritten.

Let ϵ_μ (resp. ϵ_{λ_i}) be positive such that $\mu+\epsilon_\mu\in[\mu_l,\mu_u]$ (resp. $\lambda_i+\epsilon_{\lambda_i}\in[\lambda_l,\lambda_u]$).

- For all state space $\mathcal{X}$ and for all direction $\mathbf{a}$, $\mathcal{T}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ without conditions.
- $\mathtt{IM}_{\boldsymbol{\epsilon}}(\mathcal{O}_0)$ is positive if v is $\mathtt{S}_{\boldsymbol{\epsilon},-\mathbf{e}}$ which is true because ϵ_μ is positive. (resp. $\mathtt{IM}_{\boldsymbol{\epsilon}}(\mathcal{O}_{i>0})$ is positive if v is $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}}$ which is true because ϵ_{λ_i} is positive.)

So $v^\star(\mathbf{x})$ is convex in $\mu\in[\mu_l,\mu_u]$ resp. $\lambda_i\in[\lambda_l,\lambda_u]$) if the optimal thresholds S_i^* do not change on the set $[\mu_l,\mu_u]$ (resp. $[\lambda_l,\lambda_u]$).

D.2.2. Effect of h and c_i : concavity and piecewise linearity. With $\epsilon_h \geq 0$ and $\epsilon_{c_i} \leq 0$, v is $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}}$ and operators $\mathcal{C}$ (with $\mathbf{a}=\mathbf{e}$) and $\mathcal{T}$ (with $\mathbf{a}=-\mathbf{e}$) propagate $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$. So v is concave in ϵ_h and ϵ_c.

We consider a set of parameters $[h_l,h_u]$ (resp. $[c_l,c_u]$) such that the optimal thresholds S_i^* do not change on this set. As previously the MDP formulation can be rewritten on this set with translation operator only.

With $\epsilon_h \geq 0$ (resp. $\epsilon_{c_i} \geq 0$) such that $h+\epsilon_h \in [h_l,h_u]$ (resp. $c_i+\epsilon_{c_i} \in [c_l,c_u]$), then $\mathcal{T}$ propagates $\mathtt{S}^{ub}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ and $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ without conditions $\forall \mathcal{X}$ and $\forall \mathbf{a}$.

Given that v $\mathtt{S}^{ub}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ and $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ imply that v is linear in $\boldsymbol{\epsilon}$, the optimal value function $v^\star(\mathbf{x})$ is linear in $h \in [h_l,h_u]$ (resp. $c_i \in [c_l,c_u]$) if the optimal thresholds S_i^* do not change on the set $[h_l,h_u]$ (resp. $[c_l,c_u]$).

Appendix E: Tandem queue, proof of Theorem 3

The optimality equations for the tandem queue problem are

$$\begin{aligned}
\boldsymbol{\mathcal{M}}v &= \mathcal{H}+\mu_1\mathcal{O}_1 v+\mu_2\mathcal{O}_2 v+\lambda\mathcal{O}_3 v+p_0 v,\\
\mathcal{H}(\mathbf{x}) &= h_1x_1+h_2\max\{x_2,0\}+b\max\{-x_2,0\},\\
\mathcal{O}_1 v(\mathbf{x}) &= \mathcal{C}v(\mathbf{x}) \text{ with } \begin{cases}\mathbf{a}=\mathbf{e}_1,\mathbf{b}=\mathbf{0},\\ c_a=c_r=0,\end{cases}\\
\mathcal{O}_2 v(\mathbf{x}) &= \mathcal{C}v(\mathbf{x}) \text{ with } \begin{cases}\mathbf{a}=\mathbf{e}_2-\mathbf{e}_1,\mathbf{b}=\mathbf{0},\\ c_a=c_b=c_r=0,\end{cases}\\
\mathcal{O}_3 v(\mathbf{x}) &= \mathcal{T}v(\mathbf{x}) \text{ with } \begin{cases}\mathbf{a}=-\mathbf{e}_2,\mathbf{b}=\mathbf{0},\\ c_a=c_b=c_r=0.\end{cases}
\end{aligned}$$

From Veatch and Wein (1992) we know that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}$, and $\mathtt{S}_{\mathbf{e}_2,\mathbf{e}_2-\mathbf{e}_1}$.

E.1. Monotonicity

We look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{M}}$ that propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$. From Proposition 2 we obtain that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ if the following condition is satisfied, knowing that v is $\mathtt{I}_{\boldsymbol{\epsilon}}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}$, and $\mathtt{S}_{\mathbf{e}_2,\mathbf{e}_2-\mathbf{e}_1}$.

$$\begin{aligned}
&\left|\Delta_{\boldsymbol{\epsilon}}(h_1x_1+h_2x_2^+ + b(-x_2)^+)\geq 0\right|,\\
&\bigwedge\left[\begin{matrix}|\mathcal{O}_1 \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}|\\ \bigwedge\begin{pmatrix}|\epsilon_{\mu_1}<0|\wedge|\Omega_{\mathcal{O}_1}v\leq 0|\\ \vee|\epsilon_{\mu_1}>0|\wedge|\Omega_{\mathcal{O}_1}v\geq 0|\\ \vee|\epsilon_{\mu_1}=0|\end{pmatrix}\end{matrix}\right]\bigwedge\left[\begin{matrix}|\mathcal{O}_2 \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}|\\ \bigwedge\begin{pmatrix}|\epsilon_{\mu_2}<0|\wedge|\Omega_{\mathcal{O}_2}v\leq 0|\\ \vee|\epsilon_{\mu_2}>0|\wedge|\Omega_{\mathcal{O}_2}v\geq 0|\\ \vee|\epsilon_{\mu_2}=0|\end{pmatrix}\end{matrix}\right]\\
&\bigwedge\left[\begin{matrix}|\mathcal{O}_3 \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}|\\ \bigwedge\begin{pmatrix}|\epsilon_{\lambda}<0|\wedge|\Omega_{\mathcal{O}_3}v\leq 0|\\ \vee|\epsilon_{\lambda}>0|\wedge|\Omega_{\mathcal{O}_3}v\geq 0|\\ \vee|\epsilon_{\lambda}=0|\end{pmatrix}\end{matrix}\right]\bigwedge\begin{pmatrix}|\epsilon_\eta<0|\wedge|v \text{ is } \mathtt{P}|\\ \vee|\epsilon_\eta>0|\wedge|v \text{ is } \mathtt{N}|\\ \vee|\epsilon_\eta=0|\end{pmatrix}.
\end{aligned}\tag{16}$$

From Table 4 we obtain the following relations.

- $\left|\Delta_{\boldsymbol{\epsilon}}(h_1x_1+h_2x_2^+ +b(-x_2)^+)\geq 0\right| = |\epsilon_{h_1}\geq 0|\wedge|\epsilon_{h_2}\geq 0|\wedge|\epsilon_b\geq 0|$,
- $|\mathcal{O}_1$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}| = true$ (see cell 6).
- $|\Omega_{\mathcal{O}_1}v\leq 0| = true$ (see cell 16).
- $|\Omega_{\mathcal{O}_1}v\geq 0| = |\Delta_{\mathbf{e}_1}v\geq 0| = false$ (see cell 14).
- $|\mathcal{O}_2$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}| = true$ (see cell 6).
- $|\Omega_{\mathcal{O}_2}v\leq 0| = true$ (see cell 16).
- $|\Omega_{\mathcal{O}_2}v\geq 0| = |\Delta_{\mathbf{e}_2-\mathbf{e}_1}v\geq 0|$ false when $h_1\leq h_2$ (see cell 14).
- $|\mathcal{O}_3$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}| = true$ (see cell 5).
- $|\Omega_{\mathcal{O}_3}v\leq 0| = |\Delta_{-\mathbf{e}_2}v\leq 0| = false$ (see cell 15).
- $|\Omega_{\mathcal{O}_3}v\geq 0| = |\Delta_{-\mathbf{e}_2}v\geq 0| = false$ (see cell 13).
- $|v$ is $\mathtt{P}| = true$ because all costs are positive.
- $|v$ is $\mathtt{N}| = false$ because all costs are positive.

So equation (16) can be reduced, and $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ if

$$|\epsilon_{h_1}\geq 0|\wedge|\epsilon_{h_2}\geq 0|\wedge|\epsilon_b\geq 0|\wedge|\epsilon_{\mu_1}\leq 0|\wedge|\epsilon_{\mu_2}\leq 0|\wedge|\epsilon_\lambda=0|\wedge|\epsilon_\eta\leq 0|\,. \tag{17}$$

Conclusion, the optimal value function is increasing in the costs h_i and b, and decreasing in the service rate μ_i and the discount rate η.

E.2. Convexity/concavity

First we look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{M}}$ that propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$. However $|\mathcal{O}_1$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}| = false$, so $\boldsymbol{\mathcal{M}}$ does not propagate $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ (see Proposition 3 and cell 10 in Table 4).

Now we look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{M}}$ that propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$. From Proposition 3 we obtain that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ if the following condition is satisfied, knowing that v is $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}$, and $\mathtt{S}_{\mathbf{e}_2,\mathbf{e}_2-\mathbf{e}_1}$.

$$\begin{aligned}
&\left|\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}(h_1x_1+h_2x_2^+ +b(-x_2)^+)\leq 0\right| \\
&\bigwedge\left[\begin{array}{c}|\mathcal{O}_1 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| \\ \bigwedge\begin{pmatrix}|\epsilon_{\mu_1}>0|\wedge|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_1}v\leq 0| \\ \vee|\epsilon_{\mu_1}<0|\wedge|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_1}v\geq 0| \\ \vee|\epsilon_{\mu_1}=0|\end{pmatrix}\end{array}\right]\bigwedge\left[\begin{array}{c}|\mathcal{O}_2 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| \\ \bigwedge\begin{pmatrix}|\epsilon_{\mu_2}>0|\wedge|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_2}v\leq 0| \\ \vee|\epsilon_{\mu_2}<0|\wedge|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_2}v\geq 0| \\ \vee|\epsilon_{\mu_2}=0|\end{pmatrix}\end{array}\right] \\
&\bigwedge\left[\begin{array}{c}|\mathcal{O}_3 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| \\ \bigwedge\begin{pmatrix}|\epsilon_{\lambda}>0|\wedge|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_3}v\leq 0| \\ \vee|\epsilon_{\lambda}<0|\wedge|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_3}v\geq 0| \\ \vee|\epsilon_{\lambda}=0|\end{pmatrix}\end{array}\right]\bigwedge\begin{pmatrix}|\epsilon_\eta>0|\wedge|v \text{ is } \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \vee|\epsilon_\eta<0|\wedge|v \text{ is } \mathtt{I}_{-\boldsymbol{\epsilon}}| \\ \vee|\epsilon_\eta=0|\end{pmatrix}.
\end{aligned} \tag{18}$$

From Table 4 we obtain the following relations.

- $\left|\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}(h_1x_1+h_2x_2^+ +b(-x_2)^+)\leq 0\right| = true$,
- $|\mathcal{O}_1$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| = \mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}\vee\mathtt{S}^{ub}_{\mathbf{e}_1,\boldsymbol{\epsilon}}$ (see cell 10),

- $|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_1}v\leq 0|=\mathtt{S}_{-\boldsymbol{\epsilon},\mathbf{e}_1}$ (see cell 18),
- $|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_1}v\geq 0|=\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$ (see cell 18),
- $|\mathcal{O}_2$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}|=\mathtt{S}_{\mathbf{e}_2-\mathbf{e}_1,\boldsymbol{\epsilon}}\vee\mathtt{S}^{ub}_{\mathbf{e}_2-\mathbf{e}_1,\boldsymbol{\epsilon}}$ (see cell 10),
- $|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_2}v\leq 0|=\mathtt{S}_{-\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}$ (see cell 18),
- $|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_2}v\geq 0|=\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}$ (see cell 18),
- $|\mathcal{O}_3$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}|=\mathtt{S}_{-\mathbf{e}_2,\boldsymbol{\epsilon}}\vee\mathtt{S}^{ub}_{-\mathbf{e}_2,\boldsymbol{\epsilon}}$ (see cell 9),
- $|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_3}v\leq 0|=\mathtt{S}_{-\boldsymbol{\epsilon},-\mathbf{e}_2}$ (see cell 17),
- $|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_3}v\geq 0|=\mathtt{S}_{\boldsymbol{\epsilon},-\mathbf{e}_2}$ (see cell 17),
- $|v$ is $\mathtt{I}_{\boldsymbol{\epsilon}}|$ (see equation 17). $|\epsilon_{h_1}\geq 0|\wedge|\epsilon_{h_2}\geq 0|\wedge|\epsilon_b\geq 0|\wedge|\epsilon_{\mu_1}\leq 0|\wedge|\epsilon_{\mu_2}<0|\wedge|\epsilon_\lambda=0|\wedge|\epsilon_\eta\leq 0|$.
- $|v$ is $\mathtt{I}_{-\boldsymbol{\epsilon}}|$ if (see equation 17) $|\epsilon_{h_1}\leq 0|\wedge|\epsilon_{h_2}\leq 0|\wedge|\epsilon_b\leq 0|\wedge|\epsilon_{\mu_1}\geq 0|\wedge|\epsilon_{\mu_2}\geq 0|\wedge|\epsilon_\lambda=0|\wedge|\epsilon_\eta\geq 0|$.

In the following section (see equation 15) we will see that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$, $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}$, and $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2}$ if

$$|\epsilon_{h_1}=0|\wedge|\epsilon_{h_2}\geq 0|\wedge|\epsilon_b\geq 0|\wedge|\epsilon_{\mu_1}=0|\wedge|\epsilon_{\mu_2}=0|\wedge|\epsilon_\lambda\leq 0|.$$

So $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ if

$$|\epsilon_{h_1}=0|\wedge|\epsilon_{h_2}\geq 0|\wedge|\epsilon_b\geq 0|\wedge|\epsilon_{\mu_1}=0|\wedge|\epsilon_{\mu_2}=0|\wedge|\epsilon_\lambda=0|.$$

Conclusion, the optimal value function is concave in the costs h_2 and b.

E.3. Monotonicity of the optimal policy

We look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{M}}$ that propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$ and $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}$. From Proposition 3 we obtain that $\boldsymbol{\mathcal{M}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}}$ if the conditions (19) and (20) are satisfied, knowing that v is $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$, $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}$, and $\mathtt{S}_{\mathbf{e}_2,\mathbf{e}_2-\mathbf{e}_1}$.

$$\begin{aligned}
&\left|\Delta_{\mathbf{e}_1}\Delta_{\boldsymbol{\epsilon}}(h_1x_1+h_2x_2^++b(-x_2)^+)\leq 0\right|\\
&\bigwedge\left[\begin{array}{c}|\mathcal{O}_1\text{ propagates }\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}|\\ \bigwedge\begin{pmatrix}|\epsilon_{\mu_1}>0|\wedge|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_1}v\leq 0|\\ \vee|\epsilon_{\mu_1}<0|\wedge|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_1}v\geq 0|\\ \vee|\epsilon_{\mu_1}=0|\end{pmatrix}\end{array}\right]\bigwedge\left[\begin{array}{c}|\mathcal{O}_2\text{ propagates }\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}|\\ \bigwedge\begin{pmatrix}|\epsilon_{\mu_2}>0|\wedge|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_2}v\leq 0|\\ \vee|\epsilon_{\mu_2}<0|\wedge|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_2}v\geq 0|\\ \vee|\epsilon_{\mu_2}=0|\end{pmatrix}\end{array}\right]\\
&\bigwedge\left[\begin{array}{c}|\mathcal{O}_3\text{ propagates }\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}|\\ \bigwedge\begin{pmatrix}|\epsilon_{\lambda}>0|\wedge|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_3}v\leq 0|\\ \vee|\epsilon_{\lambda}<0|\wedge|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_3}v\geq 0|\\ \vee|\epsilon_{\lambda}=0|\end{pmatrix}\end{array}\right]\bigwedge\begin{pmatrix}|\epsilon_\eta>0|\wedge|v\text{ is }\mathtt{I}_{\mathbf{e}_1}|\\ \vee|\epsilon_\eta<0|\wedge|v\text{ is }\mathtt{I}_{-\mathbf{e}_1}|\\ \vee|\epsilon_\eta=0|\end{pmatrix}.
\end{aligned}\tag{19}$$

From Table 4 we obtain the following relations.

- $\left|\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{e}_1}(h_1x_1+h_2x_2^++b(-x_2)^+)\leq 0\right|=|\epsilon_{h_1}\geq 0|\wedge|\epsilon_{h_2}\geq 0|\wedge|\epsilon_b\geq 0|$,
- $|\mathcal{O}_1$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}|=true$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_1}v\leq 0|=false$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_1}v\geq 0|=true$,

- $|\mathcal{O}_2 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_2}v \leq 0| = true$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_2}v \geq 0| = false$,
- $|\mathcal{O}_3 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_3}v \leq 0| = true$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_3}v \geq 0| = false$,
- $|v \text{ is } \mathtt{I}_{\mathbf{e}_1}| = false$,
- $|v \text{ is } \mathtt{I}_{-\mathbf{e}_1}| = false$.

$$\begin{aligned}
&\left|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Delta_{\boldsymbol{\epsilon}}(h_1x_1 + h_2x_2^+ + b(-x_2)^+) \leq 0\right| \\
&\bigwedge\left[\begin{array}{c} |\mathcal{O}_1 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}| \\ \bigwedge\begin{pmatrix} |\epsilon_{\mu_1} > 0| \wedge |\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_1}v \leq 0| \\ \vee |\epsilon_{\mu_1} < 0| \wedge |\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_1}v \geq 0| \\ \vee |\epsilon_{\mu_1} = 0| \end{pmatrix}\end{array}\right] \bigwedge\left[\begin{array}{c} |\mathcal{O}_2 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}| \\ \bigwedge\begin{pmatrix} |\epsilon_{\mu_2} > 0| \wedge |\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_2}v \leq 0| \\ \vee |\epsilon_{\mu_2} < 0| \wedge |\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_2}v \geq 0| \\ \vee |\epsilon_{\mu_2} = 0| \end{pmatrix}\end{array}\right] \\
&\bigwedge\left[\begin{array}{c} |\mathcal{O}_3 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}| \\ \bigwedge\begin{pmatrix} |\epsilon_{\lambda} > 0| \wedge |\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_3}v \leq 0| \\ \vee |\epsilon_{\lambda} < 0| \wedge |\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_3}v \geq 0| \\ \vee |\epsilon_{\lambda} = 0| \end{pmatrix}\end{array}\right] \bigwedge\begin{pmatrix} |\epsilon_\eta > 0| \wedge |v \text{ is } \mathtt{I}_{\mathbf{e}_2-\mathbf{e}_1}| \\ \vee |\epsilon_\eta < 0| \wedge |v \text{ is } \mathtt{I}_{\mathbf{e}_1-\mathbf{e}_2}| \\ \vee |\epsilon_\eta = 0| \end{pmatrix}.
\end{aligned} \tag{20}$$

From Table 4 we obtain the following relations.

- $\left|\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{e}_2-\mathbf{e}_1}(h_1x_1 + h_2x_2^+ + b(-x_2)^+) \leq 0\right| = |\epsilon_{h_1} \leq 0| \wedge |\epsilon_{h_2} \geq 0| \wedge |\epsilon_b \geq 0|$,
- $|\mathcal{O}_1 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_1}v \leq 0| = true$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_1}v \geq 0| = false$,
- $|\mathcal{O}_2 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_2}v \leq 0| = false$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_2}v \geq 0| = true$,
- $|\mathcal{O}_3 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_3}v \leq 0| = true$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_3}v \geq 0| = false$,
- $|v \text{ is } \mathtt{I}_{\mathbf{e}_2-\mathbf{e}_1}| = false$,
- $|v \text{ is } \mathtt{I}_{-\mathbf{e}_2-\mathbf{e}_1}| = false$.

So equations (19) and (20) reduce to

$$|\epsilon_{h_1} = 0| \wedge |\epsilon_{h_2} \geq 0| \wedge |\epsilon_b \geq 0| \wedge |\epsilon_{\mu_1} = 0| \wedge |\epsilon_{\mu_2} = 0| \wedge |\epsilon_\lambda \leq 0|.$$

Conclusion, the optimal switching curves $s_i(x_1)$ are increasing in the demand rate λ, the backlog costs b, and decreasing in the holding cost h_2.

Appendix F: Detailed tables

	$T_{A(i)}$	$T_{D(i)}$	T_{PD} $\left(\sum_k \mathbf{a}_k = -\mathbf{e}_i - \mathbf{e}_j\right)$	$T_{T(i,j)}$
$\mathtt{P}$	*true*	*true*	*true*	*true*
$\mathtt{N}$	*true*	*true*	*true*	*true*
$\mathtt{I}_{\boldsymbol{\epsilon}}$	*true*	*true*	*true*	*true*
$\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$	*true*	*true*	*true*	*true*
$\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$	*true*	*true*	*true*	*true*
$\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}$	*true*	*true*	*true*	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$
$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}$	*true*	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$
$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	*true*	*true*	*true*	*true*
$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}$	*true*	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$ (=*false* in most cases)	*true*
$\mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}}$	*true*	*true*	*true*	$\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$
$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}}$	*true*	$\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$
$\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$	*true*	*true*	*true*	*true*
$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}}$	*true*	$\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$	*true*
$\Omega_{\mathcal{O}} v \geq 0$	$\mathtt{I}_{\mathbf{e}_i}$	$\mathtt{D}_{\mathbf{e}_i}$	$\mathtt{D}_{\mathbf{e}_i}$	$\mathtt{I}_{\mathbf{e}_j-\mathbf{e}_i}$
$\Omega_{\mathcal{O}} v \leq 0$	$\mathtt{D}_{\mathbf{e}_i}$	$\mathtt{I}_{\mathbf{e}_i}$	$\mathtt{I}_{\mathbf{e}_i}$	$\mathtt{D}_{\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\boldsymbol{\epsilon}} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_i}$	$\mathtt{S}_{\boldsymbol{\epsilon},-\mathbf{e}_i}$	$\mathtt{S}_{\boldsymbol{\epsilon},-\mathbf{e}_i}$	$\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_i} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j,-\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_i+\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j,-\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j,-\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_j-\mathbf{e}_i} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,-\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_i}$	*false*	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{-\mathbf{e}_i} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_j,-\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{-\mathbf{e}_i-\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{-\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_j,-\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_j,-\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_i-\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,-\mathbf{e}_i} \wedge \mathtt{D}_{\mathbf{e}_i}$	*false*	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{I}_{\mathbf{e}_j-\mathbf{e}_i}$

Table 6 Detailed results for Arrival, Departure, Parallel Departure, and Tandem server operators

	$T_{CA(i)}$ and $T_{BA(i)}$	$T_{CD(i)}$
$\mathtt{P}$	$\lvert c \geq 0 \rvert$	$\lvert c \geq 0 \rvert$
$\mathtt{N}$	$\lvert c \leq 0 \rvert$	$\lvert c \leq 0 \rvert$
$\mathtt{I}_{\boldsymbol{\epsilon}}$	$\lvert \epsilon_c \geq 0 \rvert$	$\lvert \epsilon_c \geq 0 \rvert$
$\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \mathtt{S}^{ub}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c = 0 \rvert$	$\mathtt{S}^{ub}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c = 0 \rvert$
$\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \geq 0 \rvert \vee \mathtt{S}^{ub}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \leq 0 \rvert$	$\mathtt{S}^{ub}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \geq 0 \rvert \vee \mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \leq 0 \rvert$
$\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i} \wedge \lvert \epsilon_c \leq 0 \rvert \vee \mathtt{S}_{\mathbf{e}_i,-\mathbf{e}_i} \wedge \lvert \epsilon_c \geq 0 \rvert \vee \lvert \epsilon_c = 0 \rvert$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i} \wedge \lvert \epsilon_c \geq 0 \rvert \vee \lvert \epsilon_c = 0 \rvert$
$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \leq 0 \rvert$ $\vee \mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{2\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \geq 0 \rvert$ $\vee \mathtt{S}_{2\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \lvert \epsilon_c = 0 \rvert$	$\left(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,-\mathbf{e}_i} \right.$ $\left. \vee \mathtt{S}_{2\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge (\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}^{ub}_{\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \lvert \epsilon_c = 0 \rvert \right)$ $\wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_j} \wedge \lvert \epsilon_c \geq 0 \rvert$
$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \leq 0 \rvert$ $\vee \mathtt{S}^{ub}_{\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j+\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \geq 0 \rvert$ $\vee \mathtt{S}_{\mathbf{e}_j+\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \lvert \epsilon_c = 0 \rvert$	$\mathtt{S}_{\mathbf{e}_j,-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j+\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \leq 0 \rvert$ $\vee \mathtt{S}^{ub}_{\mathbf{e}_j,-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \geq 0 \rvert$ $\vee \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{\mathbf{e}_j+\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge (\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}^{ub}_{\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \lvert \epsilon_c = 0 \rvert$
$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j-2\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \leq 0 \rvert$ $\vee \mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \geq 0 \rvert$ $\vee \mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{\mathbf{e}_j-2\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \lvert \epsilon_c = 0 \rvert$	$\left(\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j-2\mathbf{e}_i,\boldsymbol{\epsilon}} \right.$ $\left. \vee \mathtt{S}_{\mathbf{e}_j-2\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge (\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}^{ub}_{\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \lvert \epsilon_c = 0 \rvert \right)$ $\wedge \mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \lvert 0 \geq \epsilon_c \rvert$
$\mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_i,-\mathbf{e}_i} \wedge \lvert \epsilon_c \leq 0 \rvert \vee \mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i} \wedge \lvert \epsilon_c \geq 0 \rvert \vee \lvert \epsilon_c = 0 \rvert$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i} \wedge \lvert \epsilon_c \leq 0 \rvert \vee \lvert \epsilon_c = 0 \rvert$
$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{-2\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \leq 0 \rvert$ $\vee \mathtt{S}^{ub}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \geq 0 \rvert$ $\vee \mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{-2\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \lvert \epsilon_c = 0 \rvert$	$\left(\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i} \right.$ $\left. \vee \mathtt{S}_{-2\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{-\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \lvert \epsilon_c = 0 \rvert \right)$ $\wedge \mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \leq 0 \rvert$
$\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \leq 0 \rvert$ $\vee \mathtt{S}^{ub}_{-\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_j+\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \geq 0 \rvert$ $\vee \mathtt{S}_{-\mathbf{e}_j+\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{-\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \lvert \epsilon_c = 0 \rvert$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_j+\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \leq 0 \rvert$ $\vee \mathtt{S}^{ub}_{\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \geq 0 \rvert$ $\vee \mathtt{S}_{-\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{-\mathbf{e}_j+\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge (\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}^{ub}_{\mathbf{e}_i,\boldsymbol{\epsilon}})$ $\wedge \lvert \epsilon_c = 0 \rvert$
$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \leq 0 \rvert$ $\vee \mathtt{S}^{ub}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{2\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \lvert \epsilon_c \geq 0 \rvert$ $\vee \mathtt{S}_{2\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \lvert \epsilon_c = 0 \rvert$	$\left(\mathtt{S}^{ub}_{\mathbf{e}_i-\mathbf{e}_j,-\mathbf{e}_i} \wedge \lvert \epsilon_c \geq 0 \rvert \right.$ $\left. \vee \mathtt{S}_{2\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{-\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \lvert \epsilon_c = 0 \rvert \right)$ $\wedge \mathtt{S}_{\boldsymbol{\epsilon},-\mathbf{e}_j} \wedge \lvert \epsilon_c \geq 0 \rvert$
$\Omega_{\mathcal{O}} v \geq 0$	$\lvert \Delta_{\mathbf{e}_i} v \geq -c \rvert$	$\lvert \Delta_{-\mathbf{e}_i} v \geq -c \rvert$
$\Omega_{\mathcal{O}} v \leq 0$	$true$	$true$
$\Delta_{\boldsymbol{\epsilon}} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_i} \wedge \lvert \epsilon_c \geq 0 \rvert$	$\mathtt{S}_{\boldsymbol{\epsilon},-\mathbf{e}_i} \wedge \lvert \epsilon_c \geq 0 \rvert$
$\Delta_{\mathbf{e}_i} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,-\mathbf{e}_i} \wedge \lvert \Delta_{-\mathbf{e}_i} v \geq -c \rvert$
$\Delta_{\mathbf{e}_i+\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,-\mathbf{e}_i} \wedge \lvert \Delta_{-\mathbf{e}_i} v \geq -c \rvert$
$\Delta_{\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j,-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_j-\mathbf{e}_i} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,-\mathbf{e}_i} \wedge \lvert \Delta_{-\mathbf{e}_i} v \leq -c \rvert$
$\Delta_{-\mathbf{e}_i} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i}$
$\Delta_{-\mathbf{e}_i-\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,-\mathbf{e}_i}$
$\Delta_{-\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_j,-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_i-\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,-\mathbf{e}_i} \wedge \lvert \Delta_{-\mathbf{e}_i} v \geq -c \rvert$

Table 7 Detailed results for Controlled Arrival, Batch Arrival, and Controlled Departure operators

	$T_{CT(i,j)}$	T_{CAF} $(\mathbf{a}=\mathbf{e}_i+\mathbf{e}_j)$
$\mathtt{P}$	$\lvert c\geq 0\rvert$	$\lvert c\geq 0\rvert$
$\mathtt{N}$	$\lvert c\leq 0\rvert$	$\lvert c\leq 0\rvert$
$\mathtt{I}_{\epsilon}$	$\lvert \epsilon_c\geq 0\rvert$	$\lvert \epsilon_c\geq 0\rvert$
$\mathtt{S}_{\epsilon,\epsilon}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}\wedge\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c=0\rvert$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}\wedge\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{\epsilon,-\epsilon}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\geq 0\rvert\vee\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\leq 0\rvert$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\geq 0\rvert\vee\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\leq 0\rvert$
$\mathtt{S}_{\mathbf{e}_i,\epsilon}$	$\mathtt{S}^{ub}_{\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}\wedge\mathtt{S}_{\mathbf{e}_j,\epsilon}\wedge\mathtt{S}_{\epsilon,\mathbf{e}_j}\wedge\lvert\epsilon_c\geq 0\rvert$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\leq 0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{2\mathbf{e}_i+\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\geq 0\rvert$ $\vee\mathtt{S}_{2\mathbf{e}_i+\mathbf{e}_j,\epsilon}\wedge\mathtt{S}_{\mathbf{e}_j,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon})\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}$	$\left(\begin{array}{c}\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\\ \vee\mathtt{S}_{\mathbf{e}_i,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}\vee\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon})\wedge\lvert\epsilon_c=0\rvert\end{array}\right)$ $\wedge\mathtt{S}_{\epsilon,\mathbf{e}_j}\wedge\lvert\epsilon_c\geq 0\rvert$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\lvert\epsilon_c\leq 0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\geq 0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon})\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{\mathbf{e}_j,\epsilon}$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\wedge\mathtt{S}_{\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\leq 0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\wedge\mathtt{S}_{2\mathbf{e}_j-\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\geq 0\rvert$ $\vee\mathtt{S}_{2\mathbf{e}_j-\mathbf{e}_i,\epsilon}\wedge\mathtt{S}_{\mathbf{e}_i,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}\vee\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon})\wedge\lvert\epsilon_c=0\rvert$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{-\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\leq 0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{\mathbf{e}_i+2\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\geq 0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_i+2\mathbf{e}_j,\epsilon}\wedge\mathtt{S}_{-\mathbf{e}_i,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon})\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}\wedge\lvert\epsilon_c\leq 0\rvert\vee\lvert\epsilon_c=0\rvert$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{-\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\leq 0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\geq 0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_j,\epsilon}\wedge\mathtt{S}_{-\mathbf{e}_i,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon})\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{-\mathbf{e}_i,\epsilon}$	$\left(\begin{array}{c}\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}\wedge\mathtt{S}_{-\mathbf{e}_j,\epsilon}\\ \vee\mathtt{S}_{-2\mathbf{e}_i+\mathbf{e}_j,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}\vee\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon})\wedge\lvert\epsilon_c=0\rvert\end{array}\right)$ $\wedge\mathtt{S}_{-\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\leq 0\rvert$	$\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{-2\mathbf{e}_i-\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\leq 0\rvert$ $\vee\mathtt{S}^{ub}_{-\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\geq 0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_j,\epsilon}\wedge\mathtt{S}_{-2\mathbf{e}_i-\mathbf{e}_j,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon})\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\epsilon}$	$\left(\begin{array}{c}\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\wedge\mathtt{S}_{-\mathbf{e}_j,\epsilon}\\ \vee\mathtt{S}_{-\mathbf{e}_i,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}\vee\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon})\wedge\lvert\epsilon_c=0\rvert\end{array}\right)$ $\wedge\mathtt{S}_{-\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\leq 0\rvert$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\leq 0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\lvert\epsilon_c\geq 0\rvert$ $\vee\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon})\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{-\mathbf{e}_j,\epsilon}$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\wedge\mathtt{S}_{-2\mathbf{e}_j+\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\leq 0\rvert$ $\vee\mathtt{S}^{ub}_{-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\wedge\mathtt{S}_{-\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\geq 0\rvert$ $\vee\mathtt{S}_{-\mathbf{e}_i,\epsilon}\wedge\mathtt{S}_{-2\mathbf{e}_j+\mathbf{e}_i,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon}\vee\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\epsilon})\wedge\lvert\epsilon_c=0\rvert$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{-2\mathbf{e}_j-\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\leq 0\rvert$ $\vee\mathtt{S}^{ub}_{-\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\geq 0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_i,\epsilon}\wedge\mathtt{S}_{-2\mathbf{e}_j-\mathbf{e}_i,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon})\wedge\lvert\epsilon_c=0\rvert$
$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\epsilon}$	$\mathtt{S}^{ub}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\wedge\lvert\epsilon_c\geq 0\rvert\vee\lvert\epsilon_c=0\rvert$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{-\mathbf{e}_j,\epsilon}\wedge\lvert\epsilon_c\leq 0\rvert$ $\vee\mathtt{S}^{ub}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}\wedge\mathtt{S}_{\mathbf{e}_i,\epsilon}\wedge\lvert\epsilon_c\geq 0\rvert$ $\vee\mathtt{S}_{\mathbf{e}_i,\epsilon}\wedge\mathtt{S}_{-\mathbf{e}_j,\epsilon}\wedge(\mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon}\vee\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\epsilon})\wedge\lvert\epsilon_c=0\rvert$
$\Omega_{\mathcal{O}}v\geq 0$	$\lvert\Delta_{\mathbf{e}_j-\mathbf{e}_i}v\geq -c\rvert$	$\lvert\Delta_{\mathbf{e}_i+\mathbf{e}_j}v\geq -c\rvert$
$\Omega_{\mathcal{O}}v\leq 0$	*true*	*true*
$\Delta_{\epsilon}\Omega_{\mathcal{O}}v\geq 0$	$\mathtt{S}_{\epsilon,\mathbf{e}_j-\mathbf{e}_i}\wedge\lvert\epsilon_c\geq 0\rvert$	$\mathtt{S}_{\epsilon,\mathbf{e}_i+\mathbf{e}_j}\wedge\lvert\epsilon_c\geq 0\rvert$
$\Delta_{\mathbf{e}_i}\Omega_{\mathcal{O}}v\geq 0$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}\wedge\lvert\Delta_{\mathbf{e}_j-\mathbf{e}_i}v\geq -c\rvert$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}$
$\Delta_{\mathbf{e}_i+\mathbf{e}_j}\Omega_{\mathcal{O}}v\geq 0$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\wedge\lvert\Delta_{\mathbf{e}_j-\mathbf{e}_i}v\geq -c\rvert$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}$
$\Delta_{\mathbf{e}_j}\Omega_{\mathcal{O}}v\geq 0$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}$
$\Delta_{\mathbf{e}_j-\mathbf{e}_i}\Omega_{\mathcal{O}}v\geq 0$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}$
$\Delta_{-\mathbf{e}_i}\Omega_{\mathcal{O}}v\geq 0$	$\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j}$
$\Delta_{-\mathbf{e}_i-\mathbf{e}_j}\Omega_{\mathcal{O}}v\geq 0$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}$
$\Delta_{-\mathbf{e}_j}\Omega_{\mathcal{O}}v\geq 0$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}$
$\Delta_{\mathbf{e}_i-\mathbf{e}_j}\Omega_{\mathcal{O}}v\geq 0$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}\wedge\lvert\Delta_{\mathbf{e}_j-\mathbf{e}_i}v\geq -c\rvert$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i+\mathbf{e}_j}$

Table 8 Detailed results for Controlled Tandem and Controled Arrival as Fork operators

	$T_{R(i,j)}$
$\mathtt{P}$	$\left\|c^j \geq 0\right\| \wedge \left\|c^i \geq 0\right\|$
$\mathtt{N}$	$\left\|c^j \leq 0\right\| \wedge \left\|c^i \leq 0\right\|$
$\mathtt{I}_{\boldsymbol{\epsilon}}$	$\left\|\epsilon_c^j \geq 0\right\| \wedge \left\|\epsilon_c^i \geq 0\right\|$
$\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} = \epsilon_{c^i}\|$
$\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} \geq \epsilon_{c^i}\| \vee \mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} \leq \epsilon_{c^i}\|$
$\mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{2\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \leq 0\|$ $\vee \mathtt{S}^{ub}_{\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \geq 0\|$ $\vee \mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{2\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \|\epsilon_{c^j} - \epsilon_{c^i} = 0\|$
$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \leq 0\|$ $\vee \mathtt{S}^{ub}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \geq 0\|$ $\vee \mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \|\epsilon_{c^j} - \epsilon_{c^i} = 0\|$
$\mathtt{S}_{\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \leq 0\|$ $\vee \mathtt{S}^{ub}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{2\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \geq 0\|$ $\vee \mathtt{S}_{2\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \|\epsilon_{c^j} - \epsilon_{c^i} = 0\|$
$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \leq 0\|$ $\vee \mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \geq 0\|$ $\vee \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} = 0\|$
$\mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}}$	$\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \leq 0\|$ $\vee \mathtt{S}^{ub}_{-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j-2\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \geq 0\|$ $\vee \mathtt{S}_{\mathbf{e}_j-2\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \|\epsilon_{c^j} - \epsilon_{c^i} = 0\|$
$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \leq 0\|$ $\vee \mathtt{S}^{ub}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \geq 0\|$ $\vee \mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \|\epsilon_{c^j} - \epsilon_{c^i} = 0\|$
$\mathtt{S}_{-\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{-2\mathbf{e}_j+\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \leq 0\|$ $\vee \mathtt{S}^{ub}_{-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \geq 0\|$ $\vee \mathtt{S}_{-\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge \mathtt{S}_{-2\mathbf{e}_j+\mathbf{e}_i,\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\boldsymbol{\epsilon}}) \wedge \|\epsilon_{c^j} - \epsilon_{c^i} = 0\|$
$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}}$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\boldsymbol{\epsilon}} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \leq 0\|$ $\vee \mathtt{S}^{ub}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i} \wedge \|\epsilon_{c^j} - \epsilon_{c^i} \geq 0\|$ $\vee \left\|\epsilon_{c^j - c^i} = 0\right\|$
$\Omega_{\mathcal{O}} v \geq 0$	$\left\|\Delta_{\mathbf{e}_i} v \geq -c^i\right\| \wedge \left\|\Delta_{\mathbf{e}_j} v \geq -c^j\right\|$
$\Omega_{\mathcal{O}} v \leq 0$	$\left\|\Delta_{\mathbf{e}_i} v \leq -c^i\right\| \vee \left\|\Delta_{\mathbf{e}_j} v \leq -c^j\right\|$
$\Delta_{\boldsymbol{\epsilon}} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_i} \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_j-\mathbf{e}_i} \wedge \left\|\epsilon_c^j \geq 0\right\| \wedge \left\|\epsilon_c^i \geq 0\right\|$
$\Delta_{\mathbf{e}_i} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_i,\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_i+\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_j-\mathbf{e}_i} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_j-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{-\mathbf{e}_i} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_i,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{-\mathbf{e}_i-\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{-\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$
$\Delta_{\mathbf{e}_i-\mathbf{e}_j} \Omega_{\mathcal{O}} v \geq 0$	$\mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_i} \wedge \mathtt{S}_{\mathbf{e}_i-\mathbf{e}_j,\mathbf{e}_j-\mathbf{e}_i}$

Table 9 Detailed results for Routing operator

References

Aktaran-Kalaycı, Tba, Hayriye Ayhan. 2009. Sensitivity of optimal prices to system parameters in a steady-state service facility. *European Journal of Operational Research* **193**(1) 120–128.

Aydin, Seray, Yalçin Akçay, Fikri Karaesmen. 2009. On the structural properties of a discrete-time single product revenue management problem. *Oper. Res. Lett.* **37**(4) 273–279.

Benjaafar, Saif, Jean-Philippe Gayon, Seda Tepe. 2010. Optimal control of a productioninventory system with customer impatience. *Operations Research Letters* **38**(4) 267–272.

Çil, Eren, E. Örmeci, Fikri Karaesmen. 2009. Effects of system parameters on the optimal policy structure in a class of queueing control problems. *Queueing Systems* **61** 273–304.

Gans, Noah, Sergei Savin. 2007. Pricing and capacity rationing for rentals with uncertain durations. *Management Science* **53**(3) 390–407.

Koole, Ger. 1998. Structural results for the control of queueing systems using event-based dynamic programming. *Queueing Systems* **30**(3) 323–339.

Koole, Ger. 2006. Monotonicity in markov reward and decision chains: Theory and applications. *Foundations and Trends in Stochastic Systems* **1** 176.

Ku, Cheng-Yuan, S. Jordan. 1997. Access control to two multiserver loss queues in series. *IEEE Transactions on Automatic Control* **42**(7) 1017–1023.

Lippman, Steven A. 1975. Applying a new device in the optimization of exponential queuing systems. *Operations Research* **23**(4) 687–710.

Müller, Alfred. 1997. How does the value function of a markov decision process depend on the transition probabilities? *Mathematics of Operations Research* **22**(4) 872–885.

Ozkan, Can, Fikri Karaesmen, Suleyman Ozekici. 2013. Structural properties of Markov modulated revenue management problems. *European journal of operational research* **225**(2) 324–331.

Satir, Benhur, Secil Savasaneril, Yasemin Serin. 2012. Pooling through lateral transshipments in service parts systems. *European journal of operational research* **220**(2) 370–377.

Smith, James E., Kevin F. McCardle. 2002. Structural properties of stochastic dynamic programs. *Operations Research* **50**(5) 796–809.

Stidham, Shaler Jr. 1985. Optimal control of admission to a queueing system. *IEEE Transactions on Automatic Control* **30**(8) 705– 713.

Veatch, Michael H., Lawrence M. Wein. 1992. Monotone control of queueing networks. *Queueing Systems* **12**(3) 391–408.

Veatch, Michael H., Lawrence M. Wein. 1994. Optimal control of a Two-Station tandem Production/Inventory system. *Operations Research* **42**(2) 337–350.

Weber, Richard R., Shaler Jr. Stidham. 1987. Optimal control of service rates in networks of queues. *Advances in Applied Probability* **19**(1) 202–218.

Zerhouni, H., J.-P. Gayon, Y. Frein. 2013. Control of a production-inventory system with product returns dependent on customer demand. *International Journal of Production Economics* **143**(1) 62–71.

Zhuang, Weifen, Michael Z.F. Li. 2012. Monotone optimal control for a class of markov decision processes. *European Journal of Operational Research* **217**(2) 342–350.