Stock Rationing in an $M/E_r/1$ Multi-class Make-to-Stock Queue with Backorders

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Jean-Philippe Gayon*
Francis de Véricourt†
Fikri Karaesmen‡

* Laboratoire G-SCOP, INP Grenoble, FRANCE
jean-philippe.gayon@inpg.fr

† Fuqua School of Business
Duke University
Durham, NC, USA
fdv1@mail.duke.edu

‡ Department of Industrial Engineering
Koç University
34450, Sariyer, Istanbul, TURKEY
fkaraesmen@ku.edu.tr

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Abstract

We consider a single-item make-to-stock production system. The item is demanded by several classes of customers arriving according to Poisson processes with different backorder costs. Item processing times have an Erlang distribution. This allows us to model the information on the production status in a tractable way. We show different properties of the optimal stock allocation policy for the general case and we fully characterize the optimal policy when the manager can divert the production to a salvage (or speculative) market. In particular we show that the optimal policy is a *Work-Storage Rationing* policy such that a demand is backordered when the current total amount of work in the system (including the on-hand inventory and the work in process) is at or below a fixed threshold corresponding to the class of the demand. We also provide an effective heuristic procedure to evaluate these optimal thresholds. This heuristic turns out to be very efficient for the problem without a salvage market as well.

1 Introduction

A stock and capacity allocation problem occurs when a common stock and the production capacity of a supplier must be shared among different markets/customers. Such problems remain at the heart of many supply chain management issues. For instance, delayed product differentiation often results in maintaining a stock of generic components for multiple end-products (de Véricourt, 2002). The design of supply contracts in presence of different retailers can also entail a stock allocation problem at the supplier (Cachon and Lariviere 1999). More recently, Desphande et al. (2003) provide an example of inventory rationing for the U.S. military.

Stock and capacity allocation problems are however very challenging and generally considered intractable as explained by Tsay et al. (1999), especially when customer demands can be backordered. Even when optimal allocation strategies can be characterized, they are usually hard to implement. Indeed, the supplier needs to take many dimensions into account (the inventory level, the number of waiting demands in the system, but also the current status of the production process, etc.) when deciding to allocate stock to some customers while backordering demands from others. The complexity of such problems greatly depends on the number of customers sharing the common stock (Ha 1997b), and on the nature of the production cycle time (Ha 2000).

In this paper, we consider a supplier that produces a standard item in a make-to-stock environment for several classes of customers. Demands for each class are Poisson processes and item processing times have an Erlang distribution. The supplier has a finite production capacity and has some information on the status of the current production. The customer classes have different values and generate different backorder penalties for the supplier. The objective is to minimize the expected discounted holding and backorder costs over an infinite horizon. At each time instant the optimal decision depends on the inventory level, the number of waiting demands of each class and the current production stage.

When the manager cannot sell the production surplus on a salvage market, we provide a partial characterization of the optimal stock and production strategy. We also derive a heuristic that is very efficient. This heuristic is easy to compute and to implement. It is based on a related problem where the manager can serve at any time an ample market with zero backorder cost. When such a salvage market exists, we fully characterize the optimal stock and capacity allocation strategy. Despite the complexity of the general setting, the structure of the optimal policy turns out to be simple to understand. To our knowledge, this is the first full characterization of the optimal control of an n-dimensional make-to-stock queue problem, with non-exponential production time. We also analyze the impact of the production time variability on the system performance which is numerically shown, maybe surprisingly, to be quasi-linear for both problems.

Our model with a salvage market corresponds to the situation where the supplier can divert inventory to a speculative (spot) market. In recent years, speculative markets for non-commodity items have developed rapidly. For instance Milner and Kouvelis (2002) mention that 80% of electronic component parts (e.g., memory chips) are sold through contract purchasing while the rest is diverted on a spot market. This development poses new challenges for both suppliers and buyers. For suppliers, the availability of a speculative market alleviates the risk of inventory costs due to excess production. In particular, suppliers may still conduct their main business through long term contracts with established customers but can also easily get rid off excess inventories in the speculative market (for which no backorder cost exist). The speculative market diminishes the importance of production control in the sense of shutting down / starting up or changing the rate of production. At the same time it accentuates the importance of how to allocate production between long term customers and the speculative market.

The model with a salvage market we present in this paper fits this framework where the long term customers have priority over the speculative market due to their contract. More precisely, we assume that long-term contracts with established customers specify late delivery penalty fees (backorder costs) while the spot market is represented as a salvage market with no backorder cost. Furthermore with the development of e-business marketplaces (such as Keith Melbourne of Convergence for instance, see also Kleindorfer and Wu, 2003), transactions on a spot market have become tremendously easier over the past years and we assume that excess inventory can be diverted to a spot market any time. We assume that the firm is always better off selling parts on the speculative market rather than stopping the production. The assumption that the system never stops working is also relevant when the production set up cost is very high. This is for instance the case for chemical process industries. The Saint-Gobain Company aims to continuously run its production lines for several years without a shutdown.

Stock and capacity allocation problems were first introduced in the context of inventory control. Topkis (1968) provides one of the earliest formulations of an optimal stock rationing problem for an uncapacitated system in discrete time. He analyzes a system with two classes of customers and shortage costs. Nahmias and Demmy (1981) also consider a rationing problem in an uncapacitated setting. They analyze the cost improvement under (r, Q) policies with rationing. Frank et al. (2004) propose effective heuristics for a system with two customer classes where the demands of the first class must be fully satisfied while demands of the second class can be partially satisfied. Melchiors et al. (2000) propose a performance evaluation method for critical level policies for continuous review systems under (r, Q)-type policies. Melchiors (2003) proposes an alternative rationing policy and assesses the performance of this policy for a similar system. Deshpande et. al (2003) optimize the parameters of (r, Q) policies with rationing, and analyze the benefit of applying this policy for a military logistics system.

These previous works assume uncapacitated replenishment systems (with exogenous replenishment lead times). For limited production capacity, on the other hand, queuing-based models provide a powerful framework which allows modeling explicitly the production capacity and the randomness of the supply process (see Buzacott and Shanthikumar, 1993). We follow this approach and model our system as a single server, single-product, make-to-stock queue with multiple demands as introduced by Ha (1997a, 1997b, 2000) in the stock rationing context.

Rationing strategies also appear in inventory transshipment problems, which has attract a lot of attention from researchers and practitioners recently. Zhao et al. (2004) characterize the structure of the optimal stock allocation and production strategies for a problem with two make-to-stock queues each serving a class of customers, and where inventory transshipment is allowed. Hu et al. (2004) study a similar problem where production capacity is uncertain. They also identify and explain counter intuitive behaviors that can appear in this context.

Ha (1997a) characterizes the optimal rationing and production policy of a multi-class $M/M/1$ make-to-stock queue with lost sales. He shows that there are thresholds for each customer class such that it is optimal to reject an arriving demand from a customer if the on-hand inventory is below the threshold for that customer (and to satisfy the demand with the stock otherwise). Carr and Duenyas (2000) analyze the structure of the optimal admission/sequencing policy for a related problem where demands from one class can be rejected. Lee and Hong (2003) numerically study the performance of a lost-sales system with Coxian processing times operating under critical level rationing policies.

When backorders are allowed, the problem of characterizing the optimal policy becomes significantly more difficult because the number of waiting demands has to be tracked for each customer class. For the backorder case, Ha (1997b) shows that the optimal stock and capacity allocation for two customer classes has a monotone structure. de Véricourt et al. (2002) generalize this result and provide a full characterization of the optimal stock and capacity allocation for n customer classes. The optimal policy specifies threshold levels such that it is optimal to satisfy an arriving demand from a customer if the on-hand inventory is above the threshold for that customer and to backorder the demand otherwise. These threshold levels also determine production priority for waiting demands in a simple way.

The models in Ha (1997a,b) and de Véricourt et al. (2002) assume exponential processing times. Because of the memoryless property of the exponential distribution, the supplier does not need to make decisions based on the current status of the production process. Information Technologies in real production systems can provide however constant access to information on the status of the production process. The manager can then exploit this knowledge and make more accurate inventory allocation decisions. We consider in this paper a multi-class $\mathrm{M/E}_r/1$ make-to-stock queue (with an Erlang-r processing time). We assume the supplier exactly knows the current stage (phase) of the Erlang distribution (which can also correspond to an actual stage of the production process) and therefore, the remaining number of stages to go before completion. This approach allows us to model the information on the production status in a tractable way.

In addition, Erlang distributions provide some flexibility in modeling the production process variability. de Véricourt et al. (2001) provide insights onto the benefit of stock allocation policies when the utilization rate and the relative importance of the customer classes vary. Because of the exponential assumption therein, the impact of production time variability in this comparison is not addressed. In this paper, we evaluate the performance of optimal stock rationing policies when the production time variability increases and the mean stays constant. These two features of the Erlang distribution (information on the production status and production time variability) yield insights that cannot be obtained with the exponential distribution assumption.

To our knowledge Ha (2000) is the only paper that has addressed dynamic optimality issues in stock allocation problem for the make-to-stock queue where the processing time has an Erlang distribution. He assumes lost sales and shows that a single state variable, the work storage level, can fully capture the inventory level and the status of the current production of the system. The problem reduces then to a single dimensional MDP. He then fully characterizes the optimal stock allocation policy: for each customer class there exists a work-storage threshold level at which it is optimal to reject a demand of this class.

Our model differs from Ha's (2000) in the assumption that demands are backordered. The backordering assumption is fundamental from an inventory management perspective and merits attention but it makes the analysis much more challenging for two reasons. First, as mentioned earlier, we deal with an $n+1$ dimensional state space since we need to keep track of the waiting demands of each class. Second, backorders require addressing a new type of decision which corresponds to the production allocation in presence of waiting demands from different classes. This problem does not exist when demands are lost.

When the manager cannot sell the production surplus on a salvage market, we propose some partial results for the optimal stock allocation policy. A full characterization seems however intractable. The approaches that have been successful so far in analyzing optimal policy for make-to-stock queues are all based on the propagation of convexity properties by iterating on the value function. When the state space has more than one dimension (typically two), this approach always requires the introduction of modularity properties (see for instance Ha 1996, 1997b, de Véricourt et al. 2000, 2002 or Zhao et al. 2004). It turns out that the optimal value function of the problem without a salvage market does not satisfy these modularity properties.

On the other hand, when the manager can sell the production surplus on a speculative market, we show that these modularity properties hold for the optimal control policy. In this case, the production decision is replaced by the simpler decision of diverting inventory to the spot market. Our analysis of this problem follows a decomposition technique introduced by de Véricourt et al (2002), which consists in relating an n-dimensional control problem to an $n-1$ dimensional subproblem and then iterating on the number of demand classes n. The application of this double induction (on time and on the dimension of the problem) to our case necessitates however many subtleties and adjustments, and the introduction of more complex modularity conditions. As the result, the analysis of an n-dimensional problem makes heavily use of the optimal structures of the k-dimensional subproblems, $k < n$. In de Véricourt et al. (2002) on the other hand, the iteration is mainly based on a single $n-1$ dimension subproblem.

More precisely, we show that the optimal allocation policy of the multi-class problem with a salvage market is characterized by n work-storage rationing thresholds associated to the n demand classes. The work-storage level is the total number of completed production stages that are required to produce the current on-hand inventory plus the work in progress. The optimal policy states then to backorder an arriving demand when the current work-storage level is below or at the corresponding threshold. This characterization leads to the construction of a heuristic using a geometric tail approximation. This heuristic turns out to be very efficient for the problem without a salvage market.

In the next section, we introduce the models and formulate the stock rationing problems with or without a salvage market. Some properties of the optimal policy for the problem without a salvage market are presented in Section 3. The structure of the optimal policy for the system with a salvage market is then characterized in Section 4. Based on this result, we suggest an efficient heuristic for the problem without a salvage market in Section 5 and we evaluate its performance in Section 6. We conclude the paper in Section 7.

2 Model Formulation

2.1 Problem without a Salvage Market

Consider a supplier who produces a single item at a single facility for n different classes of customers. The finished items are placed in a common stock. When the inventory is empty, demands are backordered. When it is not, an arriving demand can be either satisfied by the on-hand inventory or can be backordered. Items held in stock induce holding costs at rate h (per item per unit of time). Demands of Class i, $1 \le i \le n$, arrive according to a Poisson process with rate λ_i and have a unit backorder cost of b_i (per item per unit of time). Suppose without loss of generality that the backorder costs are ordered such that $b_1 > \ldots > b_n$, that is customer classes are ordered from the most valuable to the least valuable one. We denote by $\mathbf{b} = (b_1, \cdots, b_n)$ the n-dimensional vector of backorder costs.

The production process consists of r identical stages in series, each exponentially distributed with mean $1/r\mu$, and the manager of the system can observe the current stage of the production process. The supplier's facility is thus modelled by a single server whose processing time is r-Erlang distributed with mean $1/\mu$. In order to ensure stability of the system, we assume that $\rho = \sum_{i=1}^{n} \lambda_i/\mu < 1$ where ρ is the utilization rate of the system.

At any time instant, the manager of the system must decide whether to produce or not. When a part is completed, he also must decide between satisfying the waiting demand of a customer or increasing the on-hand inventory level. On the other hand, when the demand of a customer arrives to the system, the manager can either satisfy it with the on-hand inventory or backorder it in order to reserve the stock for future (more valuable) customers.

Let $i(t)$ be the number of stages completed by the part under current production at time t and $s(t)$ be the on-hand inventory at time t. We can aggregate $s(t)$ and $i(t)$ in a single variable $x_0(t) = s(t) + i(t)/r$. In the following, $x_0(t)$ will be referred to as the work-storage level. Furthermore, $i(t)$ and $s(t)$ can be inferred from $x_0(t)$ in the following way:

$$s(t) = \lfloor x_0(t) \rfloor \text{ and } i(t) = r\left(x_0(t) - \lfloor x_0(t) \rfloor\right)$$

where $\lfloor y \rfloor$ denotes the largest integer that is less than or equal to y. For example, if $r = 5$ and $x_0 = 2.6$, the inventory consists of two parts ($s(t) = 2$) and the third stage of production is accomplished ($i(t) = 3$). The work-storage level $x_0(t)$ takes its values in the set $\mathbb{N}_r = [x_0 | r x_0 \in \mathbb{N}]$, where $\mathbb{N}$ represents the set of non-negative integers. Let $-x_i(t)$, $1 \leq i \leq n$, be the number of backorders of Class i, $1 \leq i \leq n$, at time t. Hence we can describe exhaustively the system state with $\mathbf{x}(t) = (x_0(t), x_1(t), \cdots, x_n(t))$ and the state space is $S_n = \mathbb{N}_r \times (\mathbf{Z}^-)^n$, where $\mathbf{Z}^-$ represents the set of non-positive integers. Let $\mathbf{X}$ represent the random variable corresponding to $\mathbf{x}$.

A control policy states the action to take at any time given the current state $\mathbf{x}(t)$. We restrict the analysis to Markovian policies since the optimal policy belongs to this class (Puterman 1994). Let $\mathbf{a}^\pi(\mathbf{x}) = (a_0^\pi(\mathbf{x}), \cdots, a_n^\pi(\mathbf{x}))$ be the control associated with a policy π where $a_0^\pi(\mathbf{x})$ is the action to be followed each time a stage of production is completed

$$a_0^\pi(\mathbf{x}) = \begin{cases} 0 & \text{to allocate the produced item to the on-hand inventory} \\ & \text{(possible only when } (x_0 + 1/r) \in \mathbb{N}) \\ k & 1 \leq k \leq n, \text{ to satisfy a backordered demand of Class } k \\ & \text{(possible only when } x_k < 0 \text{ and } x_0 \geq 1 - 1/r) \\ n+1 & \text{not to produce} \\ & \text{(possible only when } x_0 \in \mathbb{N}) \end{cases} \quad (1)$$

Notice that, when $x_0 = 1 - 1/r$, there is no inventory ($s(t) = 0$) and $r-1$ stages of production are accomplished ($i(t) = r - 1$). Thus there is only one more stage of production to be done before one item is available either to satisfy one demand or to increase the inventory by one unit.

$a_k^\pi(\mathbf{x})$, $1 \le k \le n$, is a rationing action to be taken each time a demand of Class k arrives

$$a_k^\pi(\mathbf{x}) = \begin{cases} 0 & \text{to satisfy an arriving demand of Class } k \\ & \text{(possible only when } x_0 \ge 1) \\ k & \text{to backorder an arriving demand of Class } k \end{cases} \tag{2}$$

In state $\mathbf{x}$, the system incurs a cost rate

$$c(\mathbf{x}) = h\lfloor x_0 \rfloor - \sum_{i=1}^{n} b_i x_i \tag{3}$$

The objective is to find a control policy, π, which minimizes the expected discounted costs over an infinite horizon. We define the n-class problem $\mathbf{P}_n(\mu, \boldsymbol{\lambda}, h, \mathbf{b}, r, \alpha)$ given by

$$\min_{\pi} \lim_{T\to\infty} E^{\pi}_{\mathbf{x}(0)} \left[\int_0^T e^{-\alpha t} c(\mathbf{X}(t)) dt \right] \tag{4}$$

where α is the interest rate. We will also be interested in the closely related average cost case given by

$$\min_{\pi} \lim_{T\to\infty} \frac{E^{\pi}_{\mathbf{x}(0)} \left[\int_0^T c(\mathbf{x}(t)) dt \right]}{T}$$

Without loss of generality, we can rescale time by taking $r\mu + \sum_{i=1}^{n} \lambda_i + \alpha = 1$ and using uniformization (see Lippman 1975), the optimal value function v^* can be shown to satisfy the following optimality equations:

$$v^*(\mathbf{x}) = c(\mathbf{x}) + r\mu T_0 v^*(\mathbf{x}) + \sum_{k=1}^{n} \lambda_k T_k v^*(\mathbf{x}) \tag{5}$$

where the operators T_0 and T_k, $1 \le k \le n$, are

$$T_0 v(\mathbf{x}) = \begin{cases} \min\limits_{1 \le i \le n: x_i < 0} [v(\mathbf{x} + \mathbf{e}_0/r), v(\mathbf{x} + \mathbf{e}_0/r - \mathbf{e}_0 + \mathbf{e}_i)] & \text{if } x_0 \notin \mathbb{N}\ x_0 \ge 1 - 1/r \\ \min\limits_{1 \le i \le n: x_i < 0} [v(\mathbf{x}), v(\mathbf{x} + \mathbf{e}_0/r), v(\mathbf{x} + \mathbf{e}_0/r - \mathbf{e}_0 + \mathbf{e}_i)] & \text{if } x_0 \in \mathbb{N} \text{ and } x_0 > 0 \\ \min [v(\mathbf{x}), v(\mathbf{x} + \mathbf{e}_0/r)] & \text{if } x_0 = 0 \\ v(\mathbf{x} + \mathbf{e}_0/r) & \text{if } 0 < x_0 < 1 - 1/r \end{cases} \tag{6}$$

$$T_k v(\mathbf{x}) = \begin{cases} \min [v(\mathbf{x} - \mathbf{e}_0), v(\mathbf{x} - \mathbf{e}_k)] & \text{if } x_0 \ge 1 \\ v(\mathbf{x} - \mathbf{e}_k) & \text{if } x_0 < 1 \end{cases} \tag{7}$$

where $\mathbf{e}_i$, $0 \le i \le n$, is the i-th unit vector. For example, $\mathbf{e}_1$ denotes the $(n+1)$-dimensional vector $(0, 1, 0, \cdots, 0)$. Operator T_0 is associated with production action a_0^π and T_k, $1 \le$

$k \le n$, is associated with the rationing action a_k^π. We also define the operator T such that $Tv = c + r\mu T_0 v + \sum_{k=1}^{n} \lambda_k T_k v$. Notice that $\mathbf{x} + \mathbf{e}_0$ corresponds to $\mathbf{x}$ increased by one unit of stock whereas $\mathbf{x} + \mathbf{e}_0/r$ corresponds to $\mathbf{x}$ increased by one stage of production.

In addition, by introducing the change of variable $\mathbf{w} = \mathbf{x} + \mathbf{e}_0/r - \mathbf{e}_0$, operator T_0 can be simplified as follows:

$$T_0 v(\mathbf{x}) = \begin{cases} \min\limits_{1 \le i \le n: x_i < 0} [v(\mathbf{w} + \mathbf{e}_0), v(\mathbf{w} + \mathbf{e}_i)] & \text{if } x_0 \notin \mathbb{N} \text{ and } x_0 \ge 1 - 1/r \\ \min\limits_{1 \le i \le n: x_i < 0} [v(\mathbf{x}), v(\mathbf{w} + \mathbf{e}_0), v(\mathbf{w} + \mathbf{e}_i)] & \text{if } x_0 \in \mathbb{N} \text{ and } x_0 > 0 \\ \min [v(\mathbf{x}), v(\mathbf{x} + \mathbf{e}_0/r)] & \text{if } x_0 = 0 \\ v(\mathbf{x} + \mathbf{e}_0/r) & \text{if } 0 < x_0 < 1 - 1/r \end{cases} \tag{8}$$

It is also convenient to define the operators Δ_i, $0 \le i \le n+1$, for the real-valued function v such that $\Delta_i v(\mathbf{x}) = v(\mathbf{x} + \mathbf{e}_i) - v(\mathbf{x})$. We also define the operators Δ_{ij}, $1 \le i, j \le n+1$, such that $\Delta_{ij} v(\mathbf{x}) = \Delta_i v(\mathbf{x}) - \Delta_j v(\mathbf{x}) = v(\mathbf{x} + \mathbf{e}_i) - v(\mathbf{x} + \mathbf{e}_j)$. When $j > n$, we take $\Delta_{ij} v(\mathbf{x}) = \Delta_i v(\mathbf{x})$ (for instance $\Delta_{i(n+1)} v = \Delta_i v$). To simplify the notation, we will implicitly assume that $x_i < 0$ for $1 \le i \le n$ and $x_j < 0$ for $1 \le j \le n$ when we consider $\Delta_{ij} v(\mathbf{x})$ or $\Delta_i v(\mathbf{x})$ (otherwise these quantities are not defined). The number of customer classes of the underlying problem will also be implicit, when no confusion is possible.

Finally, in the rest of the paper, we will frequently refer to the class with the highest backorder cost which has backordered demands. This class is given by the following function m:

$$\forall \mathbf{x} \in S_n, m(\mathbf{x}) = \begin{cases} \min\limits_{i \in \{1, \cdots, n\}: x_i < 0} (i) & \text{if } \exists i \in \{1, \cdots, n\}, x_i < 0 \\ n+1 & \text{otherwise} \end{cases} \tag{9}$$

2.2 Problem with a Salvage Market

There are a number of situations where shutting down production may be costly and the excess inventory can be sold relatively easily. A typical example occurs when the supplier can sell the item through a spot market in addition to its main business with long-term customers who have specific contracts. This induces a slightly different stock rationing problem where production never stops but has to be allocated between inventory for regular customers and a lower priority salvage market with ample demand. Motivated by the above points, in terms of the precise model assumptions, we assume that there exists a customer class with zero backorder cost and ample demand. We also assume that the system produces all the time.

Given a problem without a salvage market $\mathbf{P}_n(\mu, \boldsymbol{\lambda}, h, \mathbf{b}, r, \alpha)$, we introduce the corresponding problem with a salvage market $\tilde{\mathbf{P}}_n(\mu, \boldsymbol{\lambda}, h, \mathbf{b}, r, \alpha)$. The control associated with a policy π is denoted for Problem $\tilde{\mathbf{P}}_n$ by $\tilde{\mathbf{a}}^\pi(\mathbf{x})$ and is defined as in (1-2), except when

$\tilde{a}_0^\pi = n+1$ which states to satisfy a demand from the salvage market (whereas $a_0^\pi = n+1$ states not to produce in the problem without a salvage market). The salvage market will be referred to as the $(n+1)$-th class of customers with zero backorder cost $b_{n+1} = 0$. The objective is still to characterize the optimal policy which minimizes the expected discounted cost as defined in (4) and the optimal value functions of Problem $\tilde{\mathbf{P}}_n$ can similarly be shown to satisfy the following optimality equations

$$\tilde{v}^*(\mathbf{x}) = c(\mathbf{x}) + r\mu\tilde{T}_0\tilde{v}^*(\mathbf{x}) + \sum_{k=1}^{n} \lambda_k \tilde{T}_k \tilde{v}^*(\mathbf{x}) \tag{10}$$

where $\tilde{T}_k = T_k$, $1 \le k \le n$, and operator $\tilde{T}_0$ is

$$\tilde{T}_0 v(\mathbf{x}) = \begin{cases} \min\limits_{1\le i\le n+1: x_i<0} [v(\mathbf{x}+\mathbf{e}_0/r), v(\mathbf{x}+\mathbf{e}_0/r - \mathbf{e}_0 + \mathbf{e}_i)] & \text{if } x_0 \ge 1-1/r \\ v(\mathbf{x}+\mathbf{e}_0/r) & \text{if } x_0 < 1-1/r. \end{cases} \tag{11}$$

where $\mathbf{e}_{n+1} = \mathbf{0}$. It is assumed that the system always produces. This is plausible since it is always possible to satisfy a demand from the salvage market. As a result, there is a term $v(\mathbf{x})$ in $T_0 v(\mathbf{x})$ which corresponds to the option of not producing in the problem without a salvage market. This term is not present in $\tilde{T}_0 v(\mathbf{x})$. On the other hand, the term $v(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_0+\mathbf{e}_{n+1})$ in $\tilde{T}_0 v(\mathbf{x})$ does not appear in $T_0 v(\mathbf{x})$ and corresponds to the decision of selling the produced part on the salvage market, .

In addition, by introducing again the change of variable $\mathbf{w} = \mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_0$, operator $\tilde{T}_0$ can be simplified as follows:

$$\tilde{T}_0 v(\mathbf{x}) = \begin{cases} \min\limits_{1\le i\le n+1: x_i<0} [v(\mathbf{w}+\mathbf{e}_0), v(\mathbf{w}+\mathbf{e}_i)] & \text{if } x_0 \ge 1-1/r \\ v(\mathbf{w}+\mathbf{e}_0) & \text{if } x_0 < 1-1/r \end{cases} \tag{12}$$

A transition from state $\mathbf{x}$ to state $\mathbf{w}$ corresponds to the allocation of a part to the salvage market. This formulation greatly facilitates the analysis.

Finally, the operators Δ_i and Δ_{ij} are still well defined as well as the function $m(\mathbf{x})$. Note that for Problem $\tilde{\mathbf{P}}_n$, $m(\mathbf{x}) = n+1$ actually designates the salvage market, since the associated backorder cost is zero.

3 A Partial Characterization of the Optimal Policy for the Problem without a Salvage Market

3.1 The Single-class Problem

We start by studying the problem with one class of demands. The single-class problem also sheds some light into the difficulties to analyze the multi-class case.

When there is only one class of customers, the problem is to decide when to satisfy demands of Class 1 and when to produce. A simple sample path argument (not detailed) shows that it is always optimal to satisfy a Class 1 demand. Therefore, we can not have both inventory and backorders of Class 1 and the state variable of the system can be described by a single variable x_0 with $\lfloor x_0 \rfloor^+ = \max(0, \lfloor x_0 \rfloor)$ the inventory level and $\lfloor x_0 \rfloor^- = -\min(0, \lfloor x_0 \rfloor)$ the number of backorders of Class 1. Whatever the sign of $\lfloor x_0 \rfloor$, the number of stages completed by the part under current production is $r(x_0 - \lfloor x_0 \rfloor)$.

To identify the optimal policy, we introduce the set of functions, $\mathcal{V}_0$, defined with the following property:

$$v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_0/r) - v(\mathbf{x} + \mathbf{e}_0/r) \geq v(\mathbf{x} + \mathbf{e}_0) - v(\mathbf{x}) \quad (13)$$

The following proposition states that operator T preserves $\mathcal{V}_0$ for the single-class problem.

Proposition 1 *If $v \in \mathcal{V}_0$, then $Tv \in \mathcal{V}_0$*

Proof: See Appendix A

Using value iteration and Proposition 1, we obtain that the optimal value function belongs to $\mathcal{V}_0$. As a result, the optimal policy is of base-stock type: there exists a base-stock level S^* such that it is optimal to produce if the work-storage level x is smaller than S^* and to idle production otherwise.

3.2 The Multi-Class Problem

As expected, the multi-class problem turns out to be much more challenging than the single class problem. Nevertheless, we are able to establish a number of basic results on the structure of the optimal policy for this case. Proposition 2 establishes three basic properties described in Definition 1 for the optimal policy (where the last two are consequences of the first one).

Definition 1 *Let $\mathcal{U}_n$ be a set of functions such that $v \in \mathcal{U}_n$ if and only if*

1. $\Delta_{ij}v(\mathbf{x}) < 0$ *when* $1 \le i < j \le n$

2. $\Delta_{0j}v(\mathbf{x}) < \Delta_{0i}v(\mathbf{x})$ *when* $1 \le i < j \le n$

3. $\Delta_{0j}v(\mathbf{x} - \mathbf{e_j}) < \Delta_{0i}v(\mathbf{x} - \mathbf{e}_i)$ *when* $1 \le i < j \le n$

The following proposition states that operator T preserves $\mathcal{U}_n$.

Proposition 2 *If* $v \in \mathcal{U}_n$*, then* $Tv \in \mathcal{U}_n$

Proof: See Appendix B.

The structural properties suggested by Proposition 2 are fairly intuitive. Assume that there are backorders of classes i and j with $1 \le i < j \le n$ ($b_i > b_j$), the first property states that it is better to satisfy Class i, the more expensive one. The second property states that if increasing the inventory when there are Class i backorders in the system decreases costs, then increasing the inventory when there are Class j backorders in the system also decreases costs. The third property is symmetrical to the second one: if the policy states to satisfy an arriving demand of Class j with the on-hand inventory, it also states to satisfy the arriving demands of more expensive classes.

Even though Proposition 2 establish basic properties on how to prioritize items in a multi-class systems, a complete characterization of the optimal policy requires several additional properties which turn out to be difficult to establish by our approach. In particular, for the single-class problem Equation (13) implies that v is supermodular in the production status, i, and the inventory level, s. In order to generalize Proposition 1 to the multi-dimensional problem, more modularity properties are required to ensure that Equation (13) can be propagated. For instance, with 2 demand classes, a first step to this generalization would be to show that v is supermodular in the production status, i, and the number of waiting demands of Class 2, x_2. Unfortunately, the optimal value function does not necessarily satisfy these additional modularity properties (For example, a numerical study shows that the optimal value function for $x_0 = 9.5$, $x_2 = -1$, $r = 2$, $\mu = 1$, $\lambda_1 = 0.3$, $\lambda_2 = 0.3$, $h = 0.01$, $b_1 = 10$, $b_2 = 1$, $\alpha = 0.01$, is not supermodular in s and x_2). This means that the marginal cost of continuing production can increase in the number of waiting demands, especially when the inventory level is already high. As a result, a full characterization of the optimal policy seems difficult in the multi-class case. To mitigate this last assertion, the suggested modular properties are not necessary conditions and one could imagine other value function properties to characterize the optimal control policy.

4 Characterization of the Optimal Policy for the Problem with Salvage Market

4.1 Preliminary results

We extend now results of Proposition 2 to the case with a salvage market.

Definition 2 *Let $\tilde{\mathcal{U}}_n$ be a set of functions such that $v \in \tilde{\mathcal{U}}_n$ if and only if*

1. $\Delta_{ij} v(\mathbf{x}) < 0$ *when* $1 \le i < j \le n+1$

2. $\Delta_{0j} v(\mathbf{x}) < \Delta_{0i} v(\mathbf{x})$ *when* $1 \le i < j \le n+1$

3. $\Delta_{0j} v(\mathbf{x} - \mathbf{e_j}) < \Delta_{0i} v(\mathbf{x} - \mathbf{e}_i)$ *when* $1 \le i < j \le n+1$

Property 1 of $\tilde{\mathcal{U}}_n$ applied in $j = n+1$ states that it is better to satisfy backorders of Class i, $1 \le i \le n$, than demands from the salvage market. Remember that we take implicitly $\Delta_{i(n+1)} v(\mathbf{x}) = \Delta_i v(\mathbf{x})$ for $0 \le i \le n$.

Proposition 3 *If $v \in \tilde{\mathcal{U}}_n$, then $\tilde{T} v \in \tilde{\mathcal{U}}_n$*

The proof of Proposition 3 is similar to the proof of Proposition 2 (The only difference lies in showing that $\Delta_{ij} \tilde{T}_0 v(\mathbf{x}) < 0$ but the arguments are the same.)

A direct application of value iteration implies that the optimal value function also belongs to $\tilde{\mathcal{U}}_n$. A useful property is that for $v \in \tilde{\mathcal{U}}_n$, the operators satisfy

$$\tilde{T}_0 v(\mathbf{x}) = v(\mathbf{w} + \mathbf{e}_i) + \min\left[0, \Delta_{0i} v(\mathbf{w})\right] \text{ with } i = m(\mathbf{x}) \tag{14}$$

$$\tilde{T}_k v(\mathbf{x}) = v(\mathbf{x} - \mathbf{e}_0) + \min\left[0, \Delta_{0k} v(\mathbf{x} - \mathbf{e}_k - \mathbf{e}_0)\right] \text{ for } 1 \le k \le n \tag{15}$$

which implies that the corresponding controls $\tilde{\mathbf{a}}^{\pi}$ are entirely determined by the sign of $\Delta_{0i} v$, for $1 \le i \le n+1$.

4.2 Work-Storage Rationing Policies

Consider a particular class of policies entirely described by $n+1$ parameters, one corresponding to each type of demand. Let $z_k \in \mathbb{N}_r$ be the work-storage rationing level of Class k, $1 \le k \le n+1$, that is, all arriving demands of this type are backordered when the work-storage level is below z_k. Moreover, when a part is produced it is allocated to a backordered demand of Class k, only if the work storage level x_0 is larger than or equal to z_k. It is allocated to the stock otherwise. If some of these parameters are equal, the resource is allocated to the most expensive customer class (that is to the class $m(\mathbf{x})$ in state $\mathbf{x}$). This class of policies will be referred to as Work Storage Rationing (WR) policies. In a WR policy, the decisions depend on the production status. Definition 3 gives a formal description of WR policies.

Definition 3 *A WR policy π is characterized by a $(n+1)$-dimensional rationing level vector*

$\mathbf{z} = (z_1, \cdots, z_{n+1})$ *where* $z_1 = 1 - 1/r \leq z_2 \leq \cdots \leq z_{n+1}$ *such that*

$$\tilde{a}_0^\pi(\mathbf{x}) = \begin{cases} 0 & \text{if } x_0 < z_i \text{ and } i = m(\mathbf{x}) \\ \\ i & \text{if } x_0 \geq z_i \text{ and } i = m(\mathbf{x}) \end{cases}$$

$$\tilde{a}_k^\pi(\mathbf{x}) = \begin{cases} k & \text{if } x_0 \leq z_k \\ \\ 0 & \text{if } x_0 > z_k \text{ and } m(\mathbf{x}) \geq k \end{cases}$$

In a WR policy, demands of Class 1 are always satisfied, when inventory is available, since $z_1 = 1 - 1/r$. According to such a policy and assuming that $\mathbf{x}(t = 0) = \mathbf{0}$, the recurrent region of the space is $E_r = \left[\mathbf{x} \in S_n | x_0 \leq z_{m(\mathbf{x})}\right]$. The definition leaves the policy unspecified for $x_0 > z_k^*$ and $m(\mathbf{x}) < k$. In fact a precise definition for these states is not necessary since these states do not belong to E_r.

We claim that the optimal policy is a WR policy. We will argue inductively on the number of customer classes. The construction of the proof is based on the following key property: the optimal value function of an n-class problem is closely related to the optimal value function of a k-class problem, in the region of the state space where $x_0 \leq z_k^*$. In particular, it will be shown that for this region, the corresponding controls do not depend on the demands of classes strictly greater than k. The 0-class subproblem corresponds to a problem with the salvage market only and no other customer class. The transformation which relates a n-class problem $\tilde{\mathbf{P}}_n(\mu, \boldsymbol{\lambda}, h, \mathbf{b}, r, \alpha)$ to a $(n-1)$-class subproblem is based on the decomposition of the cost function $c(\mathbf{x})$

$$c(\mathbf{x}) = c_{n-1}(\mathbf{x}^{n-1}) - b_n \left(\lfloor x_0 \rfloor + \sum_{i=1}^{n} x_i \right) \tag{16}$$

where c_{n-1} is the cost function of the $(n-1)$-class subproblem $\tilde{\mathbf{P}}_{n-1}(\mu, \boldsymbol{\lambda}^{n-1}, h + b_n, \mathbf{b}^{n-1} - b_n \mathbf{1}_{n-1}, r)$ where $\boldsymbol{\lambda}^{n-1} = (\lambda_1, \cdots, \lambda_{n-1})$, $\mathbf{b}^{n-1} = (b_1, \cdots, b_{n-1})$ and $\mathbf{1}_{n-1} = \sum_{i=1}^{n-1} \mathbf{e}_i$. We can iterate this decomposition for $k < n$

$$c_k(\mathbf{x}^k) = c_{k-1}(\mathbf{x}^{k-1}) - (b_k - b_{k+1})\left(\lfloor x_0 \rfloor + \sum_{i=1}^{k} x_i\right) \tag{17}$$

It follows that

$$\begin{cases} c_0(\mathbf{x}^0) = (h + b_1)\lfloor x_0 \rfloor \\ c_k(\mathbf{x}^k) = (h + b_{k+1})\lfloor x_0 \rfloor - \sum_{i=1}^{k}(b_i - b_{k+1})x_k \quad \text{for } 1 \leq k \leq n-1 \end{cases} \tag{18}$$

Therefore c_k is the cost of a k-class problem $\tilde{\mathbf{P}}_k(\mu, \boldsymbol{\lambda}^k, h + b_{k+1}, \mathbf{b}^k - b_{k+1}\mathbf{1}_k, r)$.

Hence, for any n-class problem we have defined n subproblems with the number of customer classes equal to $0, 1, \cdots, n-1$ respectively. We will show that the optimal policy is a WR policy by iterating on the number of classes. To start the induction, assume that the optimal policy of any $(n-1)$-class problem is a WR policy. In particular the optimal policy π^*_{n-1} of the $(n-1)$-class subproblem defined above is a WR policy, with $\mathbf{z}^* = (z^*_1, \ldots, z^*_n)$ its rationing level vector.

Based on Policy π^*_{n-1} and its associated value function v^*_{n-1} of the $(n-1)$-class subproblem, we introduce $\tilde{\mathcal{V}}_n$, a structured set of value functions. We will use again value iteration to show that the optimal value function of the n-class problem belongs to $\tilde{\mathcal{V}}_n$. In the following definition, $[\mathbf{x}]_0$ designates the first component of vector $\mathbf{x}$.

Definition 4 *Let $\tilde{\mathcal{V}}_n \subset \tilde{\mathcal{U}}_n$ such that $v \in \tilde{\mathcal{V}}_n$ if and only if:*

1. $\Delta_{ij} v(\mathbf{x}) = \Delta_{ij} v^*_{n-1}(\mathbf{x})$ *for* $0 \leq i < j \leq n$ *and* $[\mathbf{x} + \mathbf{e}_i]_0 \leq z^*_n$

2. $\Delta_{0i} v(\mathbf{x}) \geq 0$ *for* $i = m(\mathbf{x}) < n+1$ *and* $x_0 > z_n{}^* - 1$

3. *For* $x_0 > z^*_n - 1$ *and* $m(\mathbf{x}) \geq n$

   a) $v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_0/r) - v(\mathbf{x} + \mathbf{e}_n + \mathbf{e}_0/r) \geq v(\mathbf{x} + \mathbf{e}_0) - v(\mathbf{x} + \mathbf{e}_n)$

   b) $v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_n) - v(\mathbf{x} + \mathbf{e}_n + \mathbf{e}_n) \leq v(\mathbf{x} + \mathbf{e}_0) - v(\mathbf{x} + \mathbf{e}_n)$

   c) $v(\mathbf{x} + \mathbf{e}_n + \mathbf{e}_0/r) - v(\mathbf{x} + \mathbf{e}_0/r) \geq v(\mathbf{x} + \mathbf{e}_n) - v(\mathbf{x})$

   d) $v(\mathbf{x} + \mathbf{e}_0 + \mathbf{e}_0/r) - v(\mathbf{x} + \mathbf{e}_0/r) \geq v(\mathbf{x} + \mathbf{e}_0) - v(\mathbf{x})$

   e) $v(\mathbf{x} + 2\mathbf{e}_n) - v(\mathbf{x} + \mathbf{e}_n) \geq v(\mathbf{x} + \mathbf{e}_n) - v(\mathbf{x})$

4. $\Delta_0 v(\mathbf{w}) \leq 0$ *when* $x_0 < z_{n+1} = \min[x_0 | \Delta_0 v(\mathbf{w}) > 0$ *and* $m(\mathbf{x}) = n+1]$, *where* z_{n+1} *is well defined from Condition 3.d.*

Condition 1 of the previous definition links any function $v \in \tilde{\mathcal{V}}_n$ to the optimal value function of the $(n-1)$-class subproblem, when the work storage level is below the last optimal rationing level. When the work storage level is above the optimal rationing level, Condition 2 states that it is always better to satisfy a waiting demand of the class with the highest backorder cost rather than to increase the inventory level.

Condition 3 describes monotonicity properties that the value functions must satisfy in the directions of $\mathbf{e}_0$ and $\mathbf{e}_n$, when $x_k = 0$ for $0 < k < n$. These conditions guarantee in turn that the optimal rationing decisions for the demand class with the lowest backorder cost can be described with a monotone switching curve, when no demands of other classes are waiting. As we show in Lemma 1, this switching curve is actually the straight line $x_0 = z_n^*$. More formally, Condition 3 states that $\Delta_{0n} v$ is increasing in x_0/r and decreasing in x_n, $\Delta_0 v$ is increasing in x_0/r and $\Delta_n v$ is increasing in both x_0/r and x_n. Condition 3 may be also interpreted in terms of submodularity and supermodularity. More precisely v is supermodular in $(\mathbf{u}, \mathbf{v})$, with $\mathbf{u} \neq \mathbf{v}$, if for all $\mathbf{x} \in S_n$ such that $\mathbf{x}+\mathbf{v}$, $\mathbf{x}+\mathbf{u}$ and $\mathbf{x}+\mathbf{u}+\mathbf{v}$ are in S_n, we have

$$v(\mathbf{x}+\mathbf{u}+\mathbf{v}) + v(\mathbf{x}) \geq v(\mathbf{x}+\mathbf{u}) + v(\mathbf{x}+\mathbf{v})$$

The definition of the submodularity is the same but with the opposite inequality, for more on these notions see Veatch and Wein (1992) for example. Conditions 3.a, 3.c, 3.d state then that v is supermodular in $(\mathbf{e}_0 - \mathbf{e}_n, \mathbf{e}_0/r)$, $(\mathbf{e}_0/r, \mathbf{e}_n)$, $(\mathbf{e}_0/r, \mathbf{e}_0)$ respectively. Condition 3.b states that v is submodular in $(\mathbf{e}_0 - \mathbf{e}_n, \mathbf{e}_n)$ and Condition 3.e states that v is convex in x_n. It can be shown that Conditions 3.a and 3.c imply Condition 3.d and that Conditions 3.b and 3.c imply Condition 3.e. Finally Condition 4 states to satisfy demands of the salvage market if and only if there are no other backordered demands and if $x_0 \geq z_{n+1}$.

Note that for the lost sales case studied by Ha (2000), Condition 3.d is the only modularity property that v needs to verify, since that model has only one dimension. In our case, the multi-dimensional aspect of the problem requires the value function to satisfy more conditions that are also less typical.

We will show that, for any number of customer classes, the optimal policy belongs to $\tilde{\mathcal{V}}_k$ and is a WR policy. We denote by $P(n)$ this property.

Definition 5 *We say that* $P(n)$ *is true, if for all* k*-class problems,* $k \leq n$,

1. *The optimal policy is a WR policy*

2. The optimal value function belongs to $\tilde{\mathcal{V}}_k$

We prove in Appendix C that $P(0)$ is true, noting that $\tilde{\mathcal{V}}_0$ is characterized only by Condition 3.d for the 0-class case. If we assume that $P(n-1)$ is true, then $\tilde{\mathcal{V}}_n$ is well-defined and not empty since v^*_{n-1}, the optimal policy of the $(n-1)$-class problem, belongs to $\tilde{\mathcal{V}}_n$.

Lemma 1 *If $P(n-1)$ is true, then Policy π, the associated policy to $v \in \tilde{\mathcal{V}}_n$, is a WR policy of rationing vector $\mathbf{z} = [\mathbf{z}^*, z_{n+1}]$ where z_{n+1} is defined by Condition 4 of $\tilde{\mathcal{V}}_n$.*

Proof: See Appendix D.

In order to establish the second part of $P(n)$, the following lemma states that the operator $\tilde{T}$ preserves $\tilde{\mathcal{V}}_n$.

Lemma 2 *If $P(n-1)$ is true and if $v \in \tilde{\mathcal{V}}_n$ then $\tilde{T}v \in \tilde{\mathcal{V}}_n$.*

Proof: See Appendix E.

Lemma 2 shows that the WR policies associated with $\tilde{\mathcal{V}}_n$ are preserved under the assumption that $P(n-1)$ is true. In particular, $v \in \tilde{\mathcal{V}}_n$ is constructed on the optimal value functions of the k-class subproblems, $k < n$, and the optimal rationing levels of the WR policy are constituted by the optimal rationing levels of the sub-problems. Based on Lemma 2, we can then state our main result:

Theorem 1 *For all n-class problems, the optimal policy is a WR policy with the rationing level vector $\mathbf{z}^n$. In addition, $\mathbf{z}^n$ is such that for $k < n$ its projection $\mathbf{z}^k$ is the optimal rationing level vector of the k-class sub-problem:*

$$\tilde{\mathbf{P}}_k(\mu, \boldsymbol{\lambda}^k, h + b_{k+1}, \mathbf{b}^k - b_{k+1}\mathbf{1}_k, r).$$

Proof: See Appendix F.

This result can also be interpreted in terms of switching surfaces dividing the state space in different regions for which the control is constant. Under this interpretation, Theorem 1 indicates that all switching surfaces are vertical planes defined by the equations $x_0 = z^*_k$. In particular, our result is consistent with the one of Ha (1997b) who analyzed the particular case where $n = 2$ and $r = 1$. He showed the optimal rationing decision is characterized with a monotone switching curve. Our result (when $n = 2$) stipulates that this switching curve is actually a vertical line (see also de Véricourt et al. 2002). This simplifies the policy structure in a significant manner because the precise description of a generic switching curve may require infinitely many parameters whereas the vertical line is described by a single parameter.

5 Heuristics

The results of Section 3 partially uncover the priority properties of production and stock allocation policies but do not suggest a precise policy. In Section 4, we show that a WR policy is optimal for the problem with salvage market. We propose, for the problem without a salvage market, a modified WR-policy as a heuristic policy. A Modified Work-Storage Rationing (MWR) policy is a WR-policy except that the salvage market rationing level is replaced by an (integer) base-stock level. When there are no backordered demands ($m(\mathbf{x}) = n + 1$), this modified policy states to produce if the inventory level is strictly smaller than the base-stock level, and not to produce otherwise. All the other controls are the same as those in the original WR-policy.

We have not been able to prove that an MWR-policy is optimal for the problem without a salvage market. However all our numerical experiments support this hypothesis. In addition, if $\mathbf{z}^* = (z_1^*, \cdots, z_{n+1}^*)$ is the optimal rationing vector of the problem with salvage market $\tilde{\mathbf{P}}_n(\mu, \boldsymbol{\lambda}, h, \mathbf{b}, r, \alpha)$, we systematically obtained numerically that $\mathbf{z} = (z_1^*, \cdots, z_n^*, \lfloor z_{n+1}^* \rfloor)$ is the optimal rationing vector of the problem without salvage market $\mathbf{P}_n(\mu, \boldsymbol{\lambda}, h, \mathbf{b}, r, \alpha)$. These results can be explained by the fact that both systems are governed by very similar equations.

However, due to the curse of dimensionality, it rapidly becomes impossible to obtain the optimal values of the policy parameters when the number of classes increases. In the following, we first develop a heuristic to compute the optimal policy parameters of the WR-policy for the problem with a salvage market and then suggest another heuristic to compute the optimal policy parameters of the MWR-policy for the problem without a salvage market.

To compute the policy parameters of the heuristic MWR-policy, we have been inspired by the exact algorithm found by Véricourt et al. (2002) for a $M/M/1$ make-to-stock queue. The procedure is similar but we have replaced the known geometric approximation for the queue length distribution of the $M/M/1$ queue by a geometric approximation for the $M/E_r/1$ queue. In the previous sections, we concentrated on the discounted cost problem. However, the structural properties are retained for the average cost case (Weber and Stidham, 1987). From now, we consider the average cost minimization criterion which has a simpler interpretation: the optimal average cost does not depend on the initial conditions and the optimal policy parameters do not depend on the discount factor selected which facilitates various comparisons (see Ha 1997b or Ha 2000 for a similar approach).

To get around the curse of dimensionality, we use the strong relationship between a k-class problem and a $(k-1)$-class subproblem. The essence of the approximation leading to the heuristic policy is then a successive computation of the rationing levels $z_1, \cdots, z_{k+1}$. When the rationing level vector $z_1, \cdots, z_k$ of the $(k-1)$-class subproblem and the corresponding average cost g_{k-1} have been evaluated, the next rationing level z_{k+1} and optimal average cost g_k for the k-class problem can be computed by solving a single dimensional problem. Indeed, when the work-storage level is larger than z_k, all demands are satisfied with the stock and there are no backorders in recurrent states. When the work-storage level is lower than z_k, the average cost is given by g_{k-1}. By iterating this step for each subproblem, we obtain the following algorithm to compute the parameters of the WR-policy. The full justification for this algorithm is given in Appendix G.

Heuristic 1 *Consider an n-class problem. Construct the sequences ρ_k, η_k and $\tilde{z}_k$ as follows:*

Initialize $\tilde{z}_1 = 1 - 1/r$, $\rho_1 = 1$, $\eta_0 = 0$.

For $k = 1, \ldots, n$ do,

$$\rho_k = \frac{1}{\mu} \sum_{i=1}^{k} \lambda_i$$

η_k *is the solution in the interval (0,1) of the equation* $\left(\frac{r}{r + \rho_k (1 - 1/\eta_k)} \right)^r = 1/\eta_k$

$$\tilde{z}_{k+1} = \tilde{z}_k + \log_{\eta_k} \frac{\eta_k (h + b_{k+1})}{\rho_k (h + b_k) \left[\eta_k + (1 - \eta_k) \frac{1 - \rho_{k-1}}{1 - \eta_{k-1}} \right]}$$

The heuristic rationing levels z_k, $k \geq 1$, are then given by

$$z_1 = \tilde{z}_1$$

$$z_k = \max\{1 - 1/r, \lfloor r\tilde{z}_k + 1 \rfloor / r\} \quad \text{for } k = 2, 3..., n$$

$$z_{n+1} = \lfloor r\tilde{z}_{n+1} + 1 \rfloor / r$$

The MWR heuristic for our problem, without a salvage market, is obtained by rounding-off z_{n+1} in order to obtain the base-stock level.

Let us note that the above algorithm can easily be adapted to any M/G/1 make-to-stock queue. We do not pursue this adaptation here since testing the performance of the algorithm in other settings than M/E$_r$/1 would require the understanding and the computation of the optimal policy.

6 Performance of the Heuristic Policy for the problem without salvage market

In this section, we present a summary of the results on the performance of the MWR heuristic policy for the problem without a salvage market. Although we do not report these results here, the heuristic also performs extremely well for the salvage market case.

In order to evaluate the performance of this heuristic, we compared the average cost g^* of the optimal policy of the problem with the average cost g of the heuristic policy. The average costs are computed numerically by value iteration. We denote then by $\Delta g = (g - g^*)/g^*$, the relative cost increase when using the heuristic policy instead of the optimal policy. For all instances, we observe that the optimal policy is a MWR policy characterized by rationing levels z_i^*. Another measure of the heuristic performance is then to study the rationing level differences $\Delta z_i = z_i - z_i^*$.

We have evaluated the performance of the heuristic with 1, 2 and 3 classes of customers by varying the different parameters of the problems. We have not investigated with a higher number of classes because of the curse of dimensionality. In all our numerical studies we set $\mu = 1$ and $b_n = 1$ without loss of generality.

For the single-class problem, we have tested the heuristic for 567 instances corresponding to the combinations of : $r \in \{2, 3, 4, 5, 10, 15, 20\}$, $\rho \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$, $h \in \{0.0001, 0.0005, 0.001, 0.005, 0.01, 0.05, 0.1, 0.5, 1, 5\}$. We have obtained the following results :

- For all the 567 instances, $-1 \leq \Delta z_2 \leq 0$
- For 487 instances, the heuristic finds the optimal policy (i.e. $\Delta g = 0$ and $\Delta z_2 = 0$)
- If the optimal base-stock level $z_2^* \geq 6$, then $\Delta g \leq 3\%$ (544 instances)
- If the optimal base-stock level $z_2^* \geq 12$, then $\Delta g \leq 0.35\%$ (352 instances)
- The largest error is of 47% ($z_2^* = 3$ and $z_2 = 2$).

For two classes of customers, we have tested all the 728 instances combining: $r \in \{2, 3, 5, 10\}$, $\rho \in \{0.2, 0.4, 0.6, 0.8\}$, $h \in \{0.001, 0.01, 0.1, 1\}$, $b_1/b_2 \in \{1, 5, 25, 100\}$, $\lambda_1/\lambda_2 \in \{0.2, 1, 5\}$. We have obtained the following results :

- $|\Delta z_i| \leq 1$, $i = 2, 3$ (for all 728 instances)
- The heuristic finds the optimal policy for 402 instances ($\Delta g = 0$ and $\Delta z_i = 0$ for $i = 2, 3$)

- If $z_3^* \geq 3$, $\Delta_g \leq 9\%$ (544 instances)
- If $z_3^* \geq 5$, $\Delta_g \leq 5\%$ (352 instances)
- If $z_3^* \geq 10$, $\Delta_g \leq 2\%$ (150 instances)
- If $z_3^* \geq 12$, $\Delta_g \leq 1\%$ (116 instances)
- The largest error is of 139% ($z_3^* = 0$ and $z_3 = 1$).

For three classes of customers, we have tested the 243 combinations of: $r \in \{2, 3, 5\}$, $\rho \in \{0.4, 0.6, 0.8\}$, $h \in \{0.01, 0.1, 1\}$, $b_1/b_2 \in \{2, 5, 10\}$, $b_2/b_3 \in \{2, 5, 10\}$, $\lambda_1 = \lambda_2 = \lambda_3$. We have obtained the following results:

- $|\Delta z_i| \leq 1$, $i = 2, 3, 4$ (for all 243 instances)
- The heuristic finds the optimal policy for 82 instances ($\Delta g = 0$ and $\Delta z_i = 0$ for $i = 2, 3, 4$).
- If $z_4^* \geq 5$, $\Delta g \leq 3.75\%$ (92 instances)
- If $z_4^* \geq 8$, $\Delta g \leq 1\%$ (55 instances)
- If $z_4^* \geq 10$, $\Delta g < 2\%$ (42 instances)
- The largest error is of 47% ($z_4^* = 0$ and $z_4 = 1$).

In conclusion, for 1, 2 and 3 classes of customers, the heuristic finds rationing levels and base-stock level with a maximum error of 1 for all the 1538 instances tested. The relative cost increase for using the heuristic policy is always less than 2% when the base-stock level is higher than 10. However, when the base-stock level is low, a small approximation error in the base-stock level may lead to a magnified percentage error in terms of the average cost. This situation occurs whenever the holding cost h is very high or the utilization rate ρ is very low. While this is a limitation of the heuristic, it may be argued that these situations are not the most appropriate for stock rationing, and that the absolute error in terms of cost will be relatively small since the average inventory level will be very low in such cases.

7 Conclusion and future research

In this paper, we have analyzed a stock rationing problem with several customer classes where the processing times have an Erlang distribution. The Erlang distribution assumption allows us to model the information on the production status in a tractable manner and enables modeling production time variability. We have considered two cases: (1) When the manager can sell the production surplus on a salvage market and (2) when he can not.

There are two major motivations for the problem with a salvage market. First, it corresponds to the situation where the supplier can divert inventory to a speculative market and has to allocate production between long term (contract) customers and the speculative market. Second, it is a tractable approximation of the problem without a salvage market which allowed us to develop an efficient heuristic for the problem without a salvage market. For the problem with a salvage market, we have provided a full characterization of the optimal policy by exploiting the nested structure that links a $n+1$ customer class problem to a subproblem with n classes. This is the only full characterization in a dynamic allocation problem for a multi-class make-to-stock queue with non-exponential processing times and backorders, to our knowledge. In addition, the structure of the optimal policy is fairly intuitive and easy to implement. Moreover, we have proposed an efficient heuristic evaluation of the corresponding optimal parameters. This heuristic procedure allows addressing problems with a large number of customer classes that would not be tractable otherwise.

For the problem without a salvage market, we have fully characterized the optimal policy for a single-class problem and we provided a partial characterization of the optimal policy for the multi-class problems. A full characterization, in a general setting, seems to be difficult since the approach, used to propagate modularity properties for the problem with a salvage market, does not work. Based on the findings of the problem with a salvage market, we have presented a modified heuristic which performs very well for the problems without a salvage market. Moreover, based on numerical results, we conjecture that the optimal policy is an MWR policy and that the rationing levels are equal to the optimal rationing levels of the problem with salvage market.

Finally, our results constitute a useful benchmark for systems with more general processing times than Erlang distributions. These problems can be non-Markovian and are extremely difficult to analyze since the optimal decisions should take the actual processing time into account. Even if they could be characterized, these policies would most likely be hard to implement. For the deterministic case, our heuristic procedure should already perform well as Erlang distributions approach deterministic times for large numbers of stages. For the more general case, multi-stage distributions with different exponential processing times provide a promising alternative to approximate the processing time. Our heuristics can in fact be directly extended to this case. In general, the nested approach using an $M/G/1$ approximation presented in this paper offers a tractable framework to evaluate the optimal rationing levels in multi-class make-to-stock queues with generally distributed processing times.

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A Proof of Proposition 1

For the single-class problem, the optimality equations simplifies to: $v^*(\mathbf{x}) = c(\mathbf{x}) + \mu T_0 v^*(\mathbf{x}) + \lambda_1 v^*(\mathbf{x} - \mathbf{e}_0)$. Assume that $v \in \mathcal{V}_0$. Let $\delta_0 v$ be the operator such that $\delta_0 v(\mathbf{x}) = v(\mathbf{x} + \mathbf{e}_0/r) - v(\mathbf{x})$. The quantity $S = \min[x_0 \in \mathbb{N} : \delta_0 v(\mathbf{x}) \geq 0]$ is well defined and policy π, associated to v, states to produce if the work-storage level x_0 is smaller than S and to idle production otherwise. Following the definition of S, we can rewrite the operator T_0:

$$T_0 v(\mathbf{x}) = \begin{cases} v(\mathbf{x} + \mathbf{e}_0/r) & \text{if } x_0 \notin \mathbb{N} \text{ or } x_0 < S \\ v(\mathbf{x}) & \text{if } x_0 \in \mathbb{N} \text{ and } x_0 > S \end{cases} \tag{19}$$

Then we have

$$\Delta_0 T_0 v(\mathbf{x}) = \begin{cases} \Delta_0 v(\mathbf{x} + \mathbf{e}_0/r) & \text{if } x \notin \mathbb{N} \text{ or } x_0 + 1 < S \\ v(\mathbf{x} + \mathbf{e}_0) - v(\mathbf{x} + \mathbf{e}_0/r) & \text{if } x_0 + 1 = S \\ \Delta_0 v(\mathbf{x}) & \text{if } x \in \mathbb{N} \text{ and } x_0 + 1 > S \end{cases} \tag{20}$$

and

$$\begin{aligned} &\Delta_0 T_0 v(\mathbf{x} + \mathbf{e}_0/r) - \Delta_0 T_0 v(\mathbf{x}) \\ &= \begin{cases} \Delta_0 v(\mathbf{x} + 2\mathbf{e}_0/r) - \Delta_0 v(\mathbf{x} + \mathbf{e}_0/r) \geq 0 & \text{if } x + 1 + 1/r < S \\ & \text{or if } (x \notin \mathbb{N} \text{ and } x_0 + 1/r \notin \mathbb{N}) \\ -\delta_0 v(\mathbf{x} + \mathbf{e}_0/r) \geq 0 & \text{if } x_0 + 1 + 1/r = S \\ \Delta_0 v(\mathbf{x} + 2\mathbf{e}_0/r) - \Delta_0 v(\mathbf{x} + \mathbf{e}_0/r) & \\ \qquad + \delta_0 v(\mathbf{x} + \mathbf{e}_0) \geq 0 & \text{if } x_0 + 1 = S \\ 0 & \text{if } x_0 + 1 > S \text{ and } x_0 + 1/r \in \mathbb{N} \\ \Delta_0 v(\mathbf{x} + 2\mathbf{e}_0/r) - \Delta_0 v(\mathbf{x}) \geq 0 & \text{if } x_0 + 1 > S \text{ and } x_0 \in \mathbb{N} \end{cases} \end{aligned} \tag{21}$$

The inequalities in (21) come from the definition of S and the assumption $v \in \mathcal{V}_0$, so that $T_0 v \in \mathcal{V}_0$. Since the cost function $c(.)$ also belongs to $\mathcal{V}_0$, we obtain the result.

B Proof of Proposition 2

Assume that $v \in \mathcal{U}_n$ and $1 \leq i < j \leq n$. Let us show that Tv verifies the first condition of $\mathcal{U}_n$.

First of all, we have $\Delta_{ij} c(\mathbf{x}) = b_j - b_i < 0$. Let us show now that $\Delta_{ij} T_0 v(\mathbf{x}) < 0$. To that end, we have to distinguish four cases.

1. $x_0 = 0$

$$\Delta_{ij} T_0 v(\mathbf{x}) = \min\left[v(\mathbf{x}+\mathbf{e_i}), v(\mathbf{x}+\mathbf{e_i}+\mathbf{e}_0/r)\right] - \min\left[v(\mathbf{x}+\mathbf{e_j}), v(\mathbf{x}+\mathbf{e_j}+\mathbf{e}_0/r)\right]$$

If $\min\left[v(\mathbf{x}+\mathbf{e_j}), v(\mathbf{x}+\mathbf{e_j}+\mathbf{e}_0/r)\right] = v(\mathbf{x}+\mathbf{e_j})$, then

$$\Delta_{ij} T_0 v(\mathbf{x}) \le \Delta_{ij} v(\mathbf{x}) \le 0$$

If $\min\left[v(\mathbf{x}+\mathbf{e_j}), v(\mathbf{x}+\mathbf{e_j}+\mathbf{e}_0/r)\right] = v(\mathbf{x}+\mathbf{e_j}+\mathbf{e}_0/r)$, then

$$\Delta_{ij} T_0 v(\mathbf{x}) \le \Delta_{ij} v(\mathbf{x}+\mathbf{e}_0/r) \le 0$$

Therefore $\Delta_{ij} T_0 v(\mathbf{x}) \le 0$.

2. $0 < x_0 < 1 - 1/r$

$$\Delta_{ij} T_0 v(\mathbf{x}) = \Delta_{ij} v(\mathbf{x}+\mathbf{e}_0/r) \le 0$$

3. $x_0 \in \mathbb{N}$, $x_0 > 0$
Let $m(\mathbf{x}+\mathbf{e}_i) = p$ and $m(\mathbf{x}+\mathbf{e_j}) = q$. Notice that $q \le p \le j$. We have

$$\begin{aligned}\Delta_{ij} T_0 v(\mathbf{x}) &= \min\left[v(\mathbf{x}+\mathbf{e}_i), v(\mathbf{w}+\mathbf{e}_i+\mathbf{e}_0), v(\mathbf{w}+\mathbf{e}_i+\mathbf{e_p})\right] \\ &\quad - \min\left[v(\mathbf{x}+\mathbf{e}_j), v(\mathbf{w}+\mathbf{e_j}+\mathbf{e}_0), v(\mathbf{w}+\mathbf{e_j}+\mathbf{e_q})\right]\end{aligned}$$

If $\min\left[v(\mathbf{x}+\mathbf{e}_j), v(\mathbf{w}+\mathbf{e_j}+\mathbf{e}_0), v(\mathbf{w}+\mathbf{e_j}+\mathbf{e_q})\right] = v(\mathbf{x}+\mathbf{e}_j)$, then

$$\Delta_{ij} T_0 v(\mathbf{x}) \le \Delta_{ij} v(\mathbf{x}) \le 0$$

If $\min\left[v(\mathbf{x}+\mathbf{e}_j), v(\mathbf{w}+\mathbf{e_j}+\mathbf{e}_0), v(\mathbf{w}+\mathbf{e_j}+\mathbf{e_q})\right] = v(\mathbf{w}+\mathbf{e_j}+\mathbf{e}_0)$, then

$$\Delta_{ij} T_0 v(\mathbf{x}) \le \Delta_{ij} v(\mathbf{w}+\mathbf{e}_0) \le 0$$

If $\min\left[v(\mathbf{x}+\mathbf{e}_j), v(\mathbf{w}+\mathbf{e_j}+\mathbf{e}_0), v(\mathbf{w}+\mathbf{e_j}+\mathbf{e_q})\right] = v(\mathbf{w}+\mathbf{e_j}+\mathbf{e_q})$, then

$$\Delta_{ij} T_0 v(\mathbf{x}) \le v(\mathbf{w}+\mathbf{e}_i+\mathbf{e_p}) - v(\mathbf{w}+\mathbf{e_j}+\mathbf{e_q})$$

If $p = q$, then $v(\mathbf{w}+\mathbf{e}_i+\mathbf{e_p}) - v(\mathbf{w}+\mathbf{e_j}+\mathbf{e_q}) = \Delta_{ij} v(\mathbf{w}+\mathbf{e_p}) \le 0$. If $p > q$, then $q = i$ and $v(\mathbf{w}+\mathbf{e}_i+\mathbf{e_p}) - v(\mathbf{w}+\mathbf{e_j}+\mathbf{e_q}) = \Delta_{pj} v(\mathbf{w}+\mathbf{e}_i) \le 0$ (since $p \le j$). Therefore $\Delta_{ij} T_0 v(\mathbf{x}) \le 0$.

4. $x_0 \notin \mathbb{N}$, $x_0 \ge 1 - 1/r$
With the same notations, we have

$$\begin{aligned}\Delta_{ij} T_0 v(\mathbf{x}) &= \min\left[v(\mathbf{w}+\mathbf{e}_i+\mathbf{e}_0), v(\mathbf{w}+\mathbf{e}_i+\mathbf{e_p})\right] \\ &\quad - \min\left[v(\mathbf{w}+\mathbf{e_j}+\mathbf{e}_0), v(\mathbf{w}+\mathbf{e_j}+\mathbf{e_q})\right]\end{aligned}$$

If $\min\left[v(\mathbf{w}+\mathbf{e_j}+\mathbf{e}_0), v(\mathbf{w}+\mathbf{e_j}+\mathbf{e_q})\right] = v(\mathbf{w}+\mathbf{e_j}+\mathbf{e}_0)$, then

$$\Delta_{ij}T_0v(\mathbf{x}) \leq \Delta_{ij}v(\mathbf{w}+\mathbf{e}_0) \leq 0$$

Otherwise

$$\Delta_{ij}T_0v(\mathbf{x}) \leq v(\mathbf{w}+\mathbf{e}_i+\mathbf{e_p}) - v(\mathbf{w}+\mathbf{e_j}+\mathbf{e_q}) \leq 0$$

We obtain the previous inequality using the same argument as in case 3.

Let us show now that $\Delta_{ij}T_kv(\mathbf{x}) < 0$ for $1 \leq k \leq n$.

$$\begin{aligned}\Delta_{ij}T_kv(\mathbf{x}) &= \min\left[v(\mathbf{w}+\mathbf{e}_i-\mathbf{e}_0), v(\mathbf{w}+\mathbf{e}_i-\mathbf{e}_k)\right] \\ &\quad - \min\left[v(\mathbf{w}+\mathbf{e_j}-\mathbf{e}_0), v(\mathbf{w}+\mathbf{e_j}-\mathbf{e_k})\right]\end{aligned}$$

If $\min\left[v(\mathbf{w}+\mathbf{e_j}-\mathbf{e}_0), v(\mathbf{w}+\mathbf{e_j}-\mathbf{e_k})\right] = v(\mathbf{w}+\mathbf{e_j}-\mathbf{e}_0)$, then

$$\Delta_{ij}T_kv(\mathbf{x}) \leq \Delta_{ij}v(\mathbf{w}-\mathbf{e}_0) \leq 0$$

Otherwise

$$\Delta_{ij}T_kv(\mathbf{x}) \leq \Delta_{ij}v(\mathbf{w}-\mathbf{e}_k) \leq 0$$

We conclude that $\Delta_{ij}Tv(\mathbf{x}) = \Delta_{ij}c(\mathbf{x}) + r\mu\Delta_{ij}T_0v(\mathbf{x}) + \sum_{i=1}^{n}\lambda_iT_iv(\mathbf{x}) \leq 0$ and Tv verifies the first condition of $\mathcal{U}_n$. Conditions 2 and 3 are direct consequences of Condition 1, applied respectively in $\mathbf{x}$ and in $\mathbf{x}+\mathbf{e}_0-\mathbf{e}_i-\mathbf{e}_j$. Finally $Tv \in \mathcal{U}_n$.

C Characterization of the 0-class problem optimal policy

In the 0-class problem, the optimal equations are reduced to $v^*(\mathbf{x}) + g^* = c(\mathbf{x}) + r\mu\tilde{T}_0v^*(\mathbf{x})$ with $\tilde{T}_0v(\mathbf{x}) = \min\left[v(\mathbf{w}), v(\mathbf{w}+\mathbf{e}_0)\right]$ and $c(\mathbf{x}) = h\lfloor x_0\rfloor$ and $\tilde{\mathcal{V}}_0$ is characterized only by Condition 3.d. Assume that $v \in \tilde{\mathcal{V}}_0$ and $z_1^v = \min[x_0|\Delta_0v(\mathbf{w}) > 0] = 1-1/r$. Then

$$\Delta_0\tilde{T}_0v(\mathbf{x}) = \begin{cases} 0 & \text{if } x_0 < z_1^v = 1-1/r \\ \Delta_0v(\mathbf{w}) > 0 & \text{if } x_0 \geq z_1^v = 1-1/r \end{cases}$$

Hence $\tilde{T}_0v \in \tilde{\mathcal{V}}_0$ and, as $c \in \mathcal{V}_0$, $\tilde{T}v \in \tilde{\mathcal{V}}_0$. In addition $z_1^{\tilde{T}v} = 1-1/r$. By value iteration, $v^* \in \tilde{\mathcal{V}}_0$ and $z_1^* = \min[x_0|\Delta_0v^*(\mathbf{x}) \geq 0] = 1-1/r$. Therefore π^*, the optimal policy associated with v^*, is WR. We conclude that $P(0)$ is true.

D Proof of Lemma 1

Assume that $P(n-1)$ is true. Let v^*_{n-1} be the optimal value function of the $(n-1)$-class subproblem. The production control corresponding to v^*_{n-1} is equivalent to

$$\begin{cases} \Delta_{0i} v^*_{n-1}(\mathbf{w}) \le 0 & \text{if } x_0 < z^*_i \text{ and } i = m(\mathbf{x}) \\ \Delta_{0i} v^*_{n-1}(\mathbf{w}) \ge 0 & \text{if } x_0 \ge z^*_i \text{ and } i = m(\mathbf{x}) \end{cases} \tag{22}$$

Let $v \in \tilde{\tilde{\mathcal{V}}}_n$ and π its associated policy. From (22) and Condition 1 of $\tilde{\mathcal{V}}_n$, we have for $x_0 < z^*_n$ and $m(\mathbf{x}) < n+1$

$$\begin{cases} \Delta_{0i} v(\mathbf{w}) \le 0 & \text{if } x_0 < z^*_i \text{ and } i = m(\mathbf{x}) \\ \Delta_{0i} v(\mathbf{w}) \ge 0 & \text{if } x_0 \ge z^*_i \text{ and } i = m(\mathbf{x}) \end{cases}$$

For $i = m(\mathbf{x}) < n+1$ and $x_0 \ge z^*_n$, we have $\Delta_{0i} v(\mathbf{w}) \ge 0$ from Condition 2 of $\tilde{\mathcal{V}}_n$. Assume now that $m(\mathbf{x}) = n+1$ and $x_0 < z^*_n$. Then we have

$$\begin{aligned} \Delta_{0(n+1)} v(\mathbf{w}) = \Delta_0 v(\mathbf{w}) &= \Delta_n v(\mathbf{w} + \mathbf{e}_0 - \mathbf{e}_n) + \Delta_{0n} v(\mathbf{w} - \mathbf{e}_n) \\ &= \Delta_n v(\mathbf{w} + \mathbf{e}_0 - \mathbf{e}_n) + \Delta_0 v^*_{n-1}(\mathbf{w}) \end{aligned}$$

The first term of this expression is strictly negative from the first Condition of $\tilde{\mathcal{U}}_n$ while the second one is negative from Condition 4 of $\tilde{\mathcal{V}}_{n-1}$. Then $\Delta_0 v(\mathbf{w}) \le 0$ for $m(\mathbf{x}) = n+1$ and $x_0 < z^*_n$ which implies that $z_{n+1} = \min[x_0 | \Delta_0 v(\mathbf{w}) > 0 \text{ and } m(\mathbf{x}) = n+1] \ge z^*_n$. In addition, as $\Delta_0 v$ is increasing in x_0/r from Condition 3.d of $\tilde{\mathcal{V}}_n$, we have $\Delta_0 v(\mathbf{w}) > 0$ for $x_0 \ge z_{n+1}$ and $m(\mathbf{x}) = n+1$. Finally we have

$$\begin{cases} \Delta_{0i} v(\mathbf{w}) \le 0 & \text{if } x_0 < z^*_i \text{ and } i = m(\mathbf{x}) < n+1 \\ \Delta_{0i} v(\mathbf{w}) \ge 0 & \text{if } x_0 \ge z^*_i \text{ and } i = m(\mathbf{x}) < n+1 \\ \Delta_{0n+1} v(\mathbf{w}) \le 0 & \text{if } x_0 < z_{n+1} \\ \Delta_{0n+1} v(\mathbf{w}) \ge 0 & \text{if } x_0 \ge z_{n+1} \end{cases}$$

which corresponds to the production control of a WR policy of rationing vector $\mathbf{z} = [\mathbf{z}^*, z_{n+1}]$.

The rationing control for Class k corresponding to v^*_{n-1} is equivalent to

$$\begin{cases} \Delta_{0k} v^*_{n-1}(\mathbf{x} - \mathbf{e}_k - \mathbf{e}_0) < 0 & \text{if } x_0 \le z^*_k \\ \Delta_{0k} v^*_{n-1}(\mathbf{x} - \mathbf{e}_k - \mathbf{e}_0) \ge 0 & \text{if } x_0 > z^*_k \text{ and } m(\mathbf{x}) \ge k \end{cases} \tag{23}$$

which implies with Condition 1 of $\tilde{\mathcal{V}}_n$, that for $x_0 \le z^*_n$ and for $1 \le k \le n$

$$\begin{cases} \Delta_{0k} v(\mathbf{x} - \mathbf{e}_k - \mathbf{e}_0) < 0 & \text{if } x_0 \le z^*_k \\ \Delta_{0k} v(\mathbf{x} - \mathbf{e}_k - \mathbf{e}_0) \ge 0 & \text{if } x_0 > z^*_k \text{ and } m(\mathbf{x}) \ge k \end{cases} \tag{24}$$

Assume now that $x_0 > z_n^*$. If $m(\mathbf{x}) \geq k$ then $m(\mathbf{x} - \mathbf{e}_k - \mathbf{e}_0) = k < n+1$ and Condition 2 of $\tilde{\mathcal{V}}_n$ implies that $\Delta_{0k} v(\mathbf{x} - \mathbf{e}_k - \mathbf{e}_0) \geq 0$. Therefore (23) holds also for $x_0 > z_n^*$ and (24) implies that the rationing control corresponding to v is the one of a WR policy of rationing vector $\mathbf{z} = [\mathbf{z}^*, z_{n+1}]$ and Lemma 1 is proven.

E Proof of Lemma 2

Consider $v \in \tilde{\mathcal{V}}_n$ and its associated policy π. Form Lemma 1, π is a WR policy with $\mathbf{z} = [\mathbf{z}^*, z_{n+1}]$ its rationing vector, and whose associated controls are described in Definition 3. We will prove successively that $\tilde{T}v$ verifies Condition 1, Condition 2 and Condition 3 of $\tilde{\mathcal{V}}_n$.

$\tilde{T}v$ verifies Condition 1 of $\tilde{\mathcal{V}}_n$

Let $0 \leq i < j \leq n$, $p = m(\mathbf{x} + \mathbf{e}_i)$ and $q = m(\mathbf{x} + \mathbf{e}_j)$. Notice that $1 \leq q \leq p < n+1$. First of all (18) implies that $\Delta_{ij} c(\mathbf{x}) = \Delta_{ij} c_{n-1}(\mathbf{x})$. As $v \in \tilde{\mathcal{U}}_n$, we have

$$\begin{aligned}
\Delta_{ij} \tilde{T}_0 v(\mathbf{x}) &= v(\mathbf{w} + \mathbf{e}_i + \mathbf{e}_p) + \min\left[0, \Delta_{0p} v(\mathbf{w} + \mathbf{e}_i)\right] - v(\mathbf{w} + \mathbf{e}_j + \mathbf{e}_q) \\
&\quad - \min\left[0, \Delta_{0q} v(\mathbf{w} + \mathbf{e}_j)\right] \\
&= \Delta_{ij} v(\mathbf{w} + \mathbf{e}_p) + \Delta_{pq} v(\mathbf{w} + \mathbf{e}_j) + \min\left[0, \Delta_{0p} v(\mathbf{w} + \mathbf{e}_i)\right] \\
&\quad - \min\left[0, \Delta_{0q} v(\mathbf{w} + \mathbf{e}_j)\right] \qquad (25)
\end{aligned}$$

From Proposition 2 we have also $v_{n-1}^* \in \tilde{\mathcal{U}}_{n-1}$ which implies

$$\begin{aligned}
\Delta_{ij} \tilde{T}_0 v_{n-1}^*(\mathbf{x}) &= \Delta_{ij} v_{n-1}^*(\mathbf{w} + \mathbf{e}_p) + \Delta_{pq} v_{n-1}^*(\mathbf{w} + \mathbf{e}_j) \\
&\quad + \min\left[0, \Delta_{0p} v_{n-1}^*(\mathbf{w} + \mathbf{e}_i)\right] - \min\left[0, \Delta_{0q} v_{n-1}^*(\mathbf{w} + \mathbf{e}_j)\right] \qquad (26)
\end{aligned}$$

Assume that $[\mathbf{x} + \mathbf{e}_i]_0 < z_n{}^*$. Then $[\mathbf{w} + \mathbf{e}_p + \mathbf{e}_i]_0$, $[\mathbf{w} + \mathbf{e}_j + \mathbf{e}_p]_0$, $[\mathbf{w} + \mathbf{e}_i + \mathbf{e}_0]_0$ and $[\mathbf{w} + \mathbf{e}_j + \mathbf{e}_0]_0$ are smaller than or equal to z_n^*. It implies from Condition 1 of $\tilde{\mathcal{V}}_n$ that

$$\begin{cases}
\Delta_{ij} v(\mathbf{w} + \mathbf{e}_p) = \Delta_{ij} v_{n-1}^*(\mathbf{w} + \mathbf{e}_p) \\
\Delta_{qp} v(\mathbf{w} + \mathbf{e}_j) = \Delta_{qp} v_{n-1}^*(\mathbf{w} + \mathbf{e}_j) \\
\Delta_{0p} v(\mathbf{w} + \mathbf{e}_i) = \Delta_{0p} v_{n-1}^*(\mathbf{w} + \mathbf{e}_i) \\
\Delta_{0q} v(\mathbf{w} + \mathbf{e}_j) = \Delta_{0q} v_{n-1}^*(\mathbf{w} + \mathbf{e}_j)
\end{cases} \qquad (27)$$

(25), (26) and (27) imply that $\Delta_{ij} \tilde{T}_0 v(\mathbf{x}) = \Delta_{ij} \tilde{T}_0 v_{n-1}^*(\mathbf{x})$.

Assume now that $[\mathbf{x} + \mathbf{e}_i]_0 = [\mathbf{x} + \mathbf{e}_j]_0 = z_n{}^*$. Then $[\mathbf{w} + \mathbf{e}_i]_0 = [\mathbf{w} + \mathbf{e}_j]_0 > z_n^*$. In addition $m(\mathbf{x} + \mathbf{e}_i) = p$ and $m(\mathbf{x} + \mathbf{e}_j) = q$. Condition 2 of $\tilde{\mathcal{V}}_n$ implies that $\Delta_{0p} v(\mathbf{w} + \mathbf{e}_i) \geq 0$

and $\Delta_{0q}v(\mathbf{w}+\mathbf{e}_j) \geq 0$. Therefore (25) implies that

$$\Delta_{ij}\tilde{T}_0 v(\mathbf{x}) = \Delta_{ij}v(\mathbf{w}+\mathbf{e}_p) + \Delta_{pq}v(\mathbf{w}+\mathbf{e}_j) \tag{28}$$

On the other hand, π^*_{n-1} states to produce for the most expensive demand class when $x_0 = z^*_n$ which implies that

$$\Delta_{ij}\tilde{T}_0 v^*_{n-1}(\mathbf{x}) = \Delta_{ij}v^*_{n-1}(\mathbf{w}+\mathbf{e}_p) + \Delta_{pq}v^*_{n-1}(\mathbf{w}+\mathbf{e}_j) \tag{29}$$

$[\mathbf{w}+\mathbf{e}_p+\mathbf{e}_i]_0$ and $[\mathbf{w}+\mathbf{e}_j+\mathbf{e}_p]_0$ are smaller than or equal to z^*_n. (27), (28) and (29) imply that $\Delta_{ij}\tilde{T}_0 v(\mathbf{x}) = \Delta_{ij}\tilde{T}_0 v^*_{n-1}(\mathbf{x})$.

Assume now that $[\mathbf{x}+\mathbf{e}_i]_0 = z_n{}^*$ and $[\mathbf{x}+\mathbf{e}_j]_0 < z_n{}^*$. Repeating the same arguments we obtain

$$\begin{aligned}\Delta_{ij}\tilde{T}_0 v(\mathbf{x}) &= \Delta_{ij}v(\mathbf{w}+\mathbf{e}_p) + \Delta_{pq}v(\mathbf{w}+\mathbf{e}_j) - \min\left[0, \Delta_{0q}v(\mathbf{w}+\mathbf{e}_j)\right] \\ &= \Delta_{ij}v^*_{n-1}(\mathbf{w}+\mathbf{e}_p) + \Delta_{pq}v^*_{n-1}(\mathbf{w}+\mathbf{e}_j) - \min\left[0, \Delta_{0q}v^*_{n-1}(\mathbf{w}+\mathbf{e}_j)\right] \\ &= \Delta_{ij}\tilde{T}_0 v^*_{n-1}(\mathbf{x}^{\mathbf{n-1}})\end{aligned} \tag{30}$$

(30) comes from Condition 1 of $\tilde{\mathcal{V}}_n$ where we have $[\mathbf{w}+\mathbf{e}_p+\mathbf{e}_i]_0$, $[\mathbf{w}+\mathbf{e}_j+\mathbf{e}_p]_0$ and $[\mathbf{w}+\mathbf{e}_j+\mathbf{e}_0]_0$ smaller than or equal to z^*_n. We can conclude that $\tilde{T}_0 v$ satisfies Condition 1 of $\tilde{\mathcal{V}}_n$.

Let us show now that $\tilde{T}_k v$ verifies Condition 1 of $\tilde{\mathcal{V}}_n$ for $1 \leq k \leq n$. Let $[\mathbf{x}+\mathbf{e}_i]_0 \leq z_n{}^*$. We have

$$\begin{aligned}\Delta_{ij}\tilde{T}_k v(\mathbf{x}) &= \Delta_{ij}v(\mathbf{x}-\mathbf{e}_0) + \min[0, \Delta_{0k}v(\mathbf{x}+\mathbf{e}_i-\mathbf{e}_k-\mathbf{e}_0)] \\ &\quad - \min[0, \Delta_{0k}v(\mathbf{x}+\mathbf{e}_j-\mathbf{e}_k-\mathbf{e}_0)] \\ &= \Delta_{ij}v^*_{n-1}(\mathbf{x}-\mathbf{e}_0) + \min[0, \Delta_{0k}v^*_{n-1}(\mathbf{x}+\mathbf{e}_i-\mathbf{e}_k-\mathbf{e}_0] \\ &\quad - \min[0, \Delta_{0k}v^*_{n-1}(\mathbf{x}+\mathbf{e}_j-\mathbf{e}_k-\mathbf{e}_0)] \\ &= \Delta_{ij}\tilde{T}_k v^*_{n-1}(\mathbf{x})\end{aligned} \tag{31}$$

(31) comes from Condition 1 of $\tilde{\mathcal{V}}_n$. In particular, we have $\Delta_{ij}T_n v(\mathbf{x}) = \Delta_{ij}T_n v^*_{n-1}(\mathbf{x}^{n-1}) = \Delta_{ij}v^*_{n-1}(\mathbf{x}^{n-1})$.

Note then that v^*_{n-1} verifies the optimality equation of the $(n-1)$-class problem

$$(1-\lambda_n)v^*_{n-1}(\mathbf{x}^{n-1}) = c_{n-1}(\mathbf{x}^{n-1}) + r\mu\tilde{T}_0 v^*_{n-1}(\mathbf{x}^{n-1}) + \sum_{k=1}^{n}\lambda_k\tilde{T}_k v^*_{n-1}(\mathbf{x}^{n-1}) \tag{32}$$

where the time scale factor $(1-\lambda_n)$ is a consequence of the uniformization procedure with the condition $\alpha+r\mu+\Sigma_{k=1^n}\lambda_k = 1$ kept for the subproblem. Since $\Delta_{ij}c(\mathbf{x}) = \Delta_{ij}c_{n-1}(\mathbf{x}^{n-1})$ it follows that

$$\begin{aligned}\Delta_{ij}\tilde{T}v(\mathbf{x}) &= \Delta_{ij}c_{n-1}(\mathbf{x}^{n-1}) + r\mu\Delta_{ij}\tilde{T}_0 v^*_{n-1}(\mathbf{x}^{n-1}) + \sum_{k=1}^{n-1}\lambda_k\Delta_{ij}\tilde{T}_k v^*_{n-1}(\mathbf{x}^{n-1}) + \lambda_n\Delta_{ij}v^*_{n-1}(\mathbf{x}^{n-1}) \\ &= \Delta_{ij}v^*_{n-1}(\mathbf{x}^{n-1})\end{aligned} \tag{33}$$

where (33) comes from (32) and $\tilde{T}v$ satisfies Condition 1 of $\tilde{\mathcal{V}}_n$.

$\tilde{T}v$ verifies Condition 2 of $\tilde{\mathcal{V}}_n$

We assume in all this subsection that $x_0 > z_n^* - 1$ and $i = m(\mathbf{x}) < n+1$. Let $j = m(\mathbf{x}+\mathbf{e}_i)$ and notice that $j \geq i$. First of all $\Delta_{0i}c(\mathbf{x}) = h + b_i \geq 0$. From Condition 2 of $\tilde{\mathcal{V}}_n$, $a_0^\pi(\mathbf{x}+\mathbf{e}_0) = i$ and therefore

$$\begin{aligned}\Delta_{0i}\tilde{T}_0 v(\mathbf{x}) &= v(\mathbf{w}+\mathbf{e}_0+\mathbf{e}_i) - \min[v(\mathbf{w}+\mathbf{e}_i+\mathbf{e}_0), v(\mathbf{w}+\mathbf{e}_i+\mathbf{e}_j)] \\ &= \begin{cases} 0 & \text{if } \Delta_{0j}v(\mathbf{w}+\mathbf{e}_i) \leq 0 \\ \Delta_{0j}v(\mathbf{w}+\mathbf{e}_i) \geq 0 & \text{otherwise} \end{cases}\end{aligned}$$

Thus $\tilde{T}_0 v$ verifies Condition 2 of $\tilde{\mathcal{V}}_n$.

Let $1 \leq k \leq n$, $p = \tilde{a}_k^\pi(\mathbf{x}+\mathbf{e}_0)$ and $q = \tilde{a}_k^\pi(\mathbf{x}+\mathbf{e}_i)$.

$$\Delta_{0i}\tilde{T}_k v(\mathbf{x}) = v(\mathbf{x}+\mathbf{e}_0-\mathbf{e}_p) - v(\mathbf{x}+\mathbf{e}_i-\mathbf{e}_q)$$

We can distinguish 4 possible cases

1. $p = q = 0$

$$\Delta_{0i}\tilde{T}_k v(\mathbf{x}) = \Delta_{0i}v(\mathbf{x}-\mathbf{e}_0) \geq \Delta_{0k}v(\mathbf{x}+\mathbf{e}_i-\mathbf{e}_0-\mathbf{e}_k) \geq 0$$

   The first inequality comes from Condition 3 of $\tilde{\mathcal{U}}_n$ and the second one is the consequence of $q = 0$

2. $p = q = k$

$$\Delta_{0i}\tilde{T}_k v(\mathbf{x}) = \Delta_{0i}v(\mathbf{x}-\mathbf{e}_k) \geq 0$$

   from Condition 2 of $\mathcal{V}_n$

3. $p = k$ and $q = 0$

$$\Delta_{0i}\tilde{T}_k v(\mathbf{x}) = \Delta_{0i}v(\mathbf{x}-\mathbf{e}_k) + \Delta_{0k}v(\mathbf{x}+\mathbf{e}_i-\mathbf{e}_0-\mathbf{e}_k) \geq 0$$

   The first term is positive from Condition 2 of $\tilde{\mathcal{V}}_n$ and the second term is positive since $q = 0$.

4. $p = 0$ and $q = k$

$$\Delta_{0i}\tilde{T}_k v(\mathbf{x}) = \Delta_{ki}v(\mathbf{x}-\mathbf{e}_k)\begin{cases} \geq 0 & \text{if } k \geq i \\ < 0 & \text{if } k < i \end{cases}$$

Therefore $\Delta_{0i}\tilde{T}_k v(\mathbf{x}) \geq 0$ and $\Delta_{0i}\tilde{T}v(\mathbf{x}) \geq 0$ except if the 3 next conditions are verified: $k < i$, $p = 0$ and $q = k$.

Assume now that $k < i$, $p = 0$ and $q = k$ and let us prove that $\Delta_{0i}\tilde{T}v(\mathbf{x}) \geq 0$. When $q = k$, $x_0 \leq z_k^*$ and, as $z_k^* \leq z_i^*$ $(k < i)$, we have also $x_0 \leq z_i^*$. In addition $x_0 > z_n^* - 1$ (by assumption) and $z_n^* \geq z_i^*$ $(n \geq i)$ imply that $x_0 > z_i^* - 1$. Finally $z_i^* - 1 < x_0 \leq z_i^*$. As $\Delta_{0i}c = \Delta_0 c_{i-1}$ and $\Delta_{0i}\tilde{T}_k v(\mathbf{x}) \geq 0$ when $i \geq k$, we can give a lower bound to $\Delta_{0i}\tilde{T}v$

$$\Delta_{0i}\tilde{T}v(\mathbf{x}) \geq \Delta_{0i}c(\mathbf{x}) + \sum_{k=1}^{i-1} \lambda_k \Delta_{0i}\tilde{T}_k v(\mathbf{x}) = \Delta_0 c_{i-1}(\mathbf{x}) + \sum_{k=1}^{i-1} \lambda_k \Delta_{0i}\tilde{T}_k v(\mathbf{x}) \tag{34}$$

Let us denote by v_j^* the optimal value function of the j-class sub-problem $\tilde{\mathbf{P}}_j(\mu, \boldsymbol{\lambda}^j, h + b_{i+1}, \mathbf{b}^j - b_j \mathbf{1}_i, r)$ and by π_j^* the associated optimal policy. Since $x_0 \leq z_i^*$, we can apply Condition 1 of $\tilde{\mathcal{V}}_j$ to $\Delta_{ki}v_j^*$ for $i \leq j \leq n$

$$\begin{aligned}
\Delta_{0i}\tilde{T}_k v(\mathbf{x}) = \Delta_{ki}v(\mathbf{x} - \mathbf{e}_k) &= \Delta_{ki}v_{n-1}^*(\mathbf{x} - \mathbf{e}_k) \\
&= \cdots \\
&= \Delta_{ki}v_i^*(\mathbf{x} - \mathbf{e}_k) \\
&= \Delta_k v_{i-1}^*(\mathbf{x} - \mathbf{e}_k) \\
&= \Delta_0 \tilde{T}_k v_{i-1}^*(\mathbf{x})
\end{aligned} \tag{35}$$

Let us detail the last equality. As $z_i^* - 1 < x_0 \leq z_i^*$, we have $\tilde{a}_k^{\pi_i^*}(\mathbf{x} + \mathbf{e}_0) = 0$ and $\tilde{a}_k^{\pi_i^*}(\mathbf{x}) = k$ which imply that $\Delta_0 \tilde{T}_k v_{i-1}^*(\mathbf{x}) = \Delta_k v_{i-1}^*(\mathbf{x} - \mathbf{e}_k)$. From (35) we can rewrite (34)

$$\Delta_{0i}\tilde{T}v(\mathbf{x}) \geq \Delta_0 c_{i-1}(\mathbf{x}) + \sum_{k=1}^{i-1} \lambda_k \Delta_0 \tilde{T}_k v_{i-1}^*(\mathbf{x}) \tag{36}$$

As v_{i-1}^* verifies the optimality equations of the $(i-1)$-class problem, we have

$$\left(1 - \sum_{k=i}^{n} \lambda_k\right) \Delta_0 v_{i-1}^*(\mathbf{x}) = \Delta_0 c(\mathbf{x}) + r\mu \Delta_0 \tilde{T} v_{i-1}^*(\mathbf{x}) + \sum_{k=1}^{n} \lambda_k \Delta_0 \tilde{T}_k v_{i-1}^*(\mathbf{x}) \tag{37}$$

where the factor $(1 - \sum_{k=i}^{n} \lambda_k)$ is a consequence of the uniformization procedure, assuming that the rescaling condition $r\mu + \sum_{k=0}^{n} \lambda_k = 1$ is kept for the sub-problems. Using (37), we rewrite (36)

$$\Delta_{0i}\tilde{T}v(\mathbf{x}) \geq \left(1 - \sum_{k=i}^{n} \lambda_k\right) \Delta_0 v_{i-1}^*(\mathbf{x}) - r\mu \Delta_0 \tilde{T}_0 v_{i-1}^*(\mathbf{x}) \tag{38}$$

As $m(\mathbf{x}) = i$, we have also

$$\Delta_0 \tilde{T}_0 v_{i-1}^*(\mathbf{x}) = \begin{cases} 0 & \text{if } z_i^* - 1 < x_0 < z_i^* \\ \Delta_0 v_{i-1}^*(\mathbf{w}) & \text{if } x_0 = z_i^* \end{cases} \tag{39}$$

Moreover, when $x_0 = z_i^*$, $\Delta_0 v_{i-1}^*(\mathbf{w}) \leq \Delta_0 v_{i-1}^*(\mathbf{x})$ from Condition 3.d of $\tilde{\mathcal{V}}_{i-1}$. We deduce from (38), (39) and last remark that

$$\Delta_{0i}\tilde{T}v(\mathbf{x}) \begin{cases} \geq (1 - \sum_{k=i}^{n} \lambda_k)\, \Delta_0 v_{i-1}^*(\mathbf{x}) & \text{if } z_i^* - 1 < x_0 < z_i^* \\ \geq (1 - r\mu - \sum_{k=i}^{n} \lambda_k)\, \Delta_0 v_{i-1}^*(\mathbf{x}) & \text{if } x_0 = z_i^* \end{cases}$$

When $x_0 > z_i^* - 1$, we have $\Delta_0 v_{i-1}^*(\mathbf{x}) \geq 0$ from Condition 3.d and 4 of $\tilde{\mathcal{V}}_{i-1}$. Hence $\Delta_{0i}\tilde{T}v(\mathbf{x}) \geq 0$ when $k < i$, $p = 0$ and $q = k$. We conclude that $\Delta_{0i}\tilde{T}v$ verifies Condition 2 of $\tilde{\mathcal{V}}_n$.

$\tilde{T}v$ verifies Condition 3 of $\tilde{\mathcal{V}}_n$

We assume in all this subsection that $x_0 > z_n^* - 1$ and $m(\mathbf{x}) \geq n$. First of all, 3.a and 3.c imply 3.d while 3.b and 3.c imply 3.e. So we have only to prove that $\tilde{T}v$ verifies 3.a, 3.b and 3.c. Let us show that $\Delta_{0n}\tilde{T}_0 v$ in increasing in x_0/r and is decreasing in x_n.

$$\Delta_{0n}\tilde{T}_0 v(\mathbf{x}) = \begin{cases} 0 & \text{if } x_0 < z_n^* \text{ and } m(\mathbf{x}+\mathbf{e}_n) = n \\ \Delta_{0n} v(\mathbf{w}+\mathbf{e}_n) \geq 0 & \text{if } x_0 \geq z_n^* \text{ and } m(\mathbf{x}+\mathbf{e}_n) = n \\ 0 & \text{if } x_0 < z_{n+1} \text{ and } m(\mathbf{x}+\mathbf{e}_n) = n+1 \\ \Delta_0 v(\mathbf{w}+\mathbf{e}_n) \geq 0 & \text{if } x_0 \geq z_{n+1} \text{ and } m(\mathbf{x}+\mathbf{e}_n) = n+1 \end{cases}$$

3.a and 3.d imply that $\Delta_{0n}\tilde{T}_0 v$ is increasing in x_0/r. 3.b implies that $\Delta_{0n}\tilde{T}_0 v$ is decreasing in x_n when $x_0 < z_{n+1}$. When $x_0 \geq z_{n+1}$ and $m(\mathbf{x}+\mathbf{e}_n) = n$, $\Delta_{0n}\tilde{T}_0 v$ is decreasing in x_n from 3.b. When $x_0 \geq z_{n+1}$ and $m(\mathbf{x}+\mathbf{e}_n) = n+1$, we have

$$\begin{aligned} \Delta_{0n}\tilde{T}_0 v(\mathbf{x}) - \Delta_{0n}\tilde{T}_0 v(\mathbf{x}-\mathbf{e}_n) &= \Delta_0 v(\mathbf{w}+\mathbf{e}_n) - \Delta_{0n} v(\mathbf{w}) \\ &= \Delta_n v(\mathbf{w}+\mathbf{e}_0) \leq 0 \end{aligned}$$

Therefore $\Delta_{0n}\tilde{T}_0 v$ is decreasing in x_n for $x_0 > z_n^* - 1$ and $m(\mathbf{x}) \geq n$.

Let us show that $\Delta_{0n}\tilde{T}_k v$, $1 \leq k \leq n$, is increasing in x_0/r and is decreasing in x_n.

$$\Delta_{0n}\tilde{T}_k v(\mathbf{x}) = \begin{cases} \Delta_{kn} v(\mathbf{x}-\mathbf{e}_k) = \Delta_k v_{n-1}^*(\mathbf{x}-\mathbf{e}_k) \leq 0 & \text{if } x_0 \leq z_k^* \\ \Delta_{0n} v(\mathbf{x}-\mathbf{e}_0) = \Delta_0 v_{n-1}^*(\mathbf{x}-\mathbf{e}_0) \leq 0 & \text{if } z_k^* < x_0 \leq z_n^* \\ \Delta_{0n} v(\mathbf{x}-\mathbf{e}_0) \geq 0 & \text{if } x_0 > z_n^* \end{cases}$$

Assume that $x_0 \le z_k^*$. Using Condition 1 of $\tilde{\mathcal{V}}_{n-1}$, $\tilde{\mathcal{V}}_{n-2}$, $\cdots$, $\tilde{\mathcal{V}}_{k+1}$, we obtain

$$\begin{aligned}
\Delta_k v_{n-1}^*(\mathbf{x}-\mathbf{e}_k) &= \Delta_{kk+1} v_{n-1}^*(\mathbf{x}-\mathbf{e}_k-\mathbf{e}_{k+1}) + \Delta_{k+1} v_{n-1}^*(\mathbf{x}-\mathbf{e}_{k+1}) \\
&= \Delta_k v_k^*(\mathbf{x}-\mathbf{e}_k) + \Delta_{k+1} v_{n-1}^*(\mathbf{x}-\mathbf{e}_{k+1}) \\
&= \cdots \\
&= \sum_{i=k}^{n-1} \Delta_i v_i^*(\mathbf{x}-\mathbf{e}_i) \qquad (40)
\end{aligned}$$

In addition $\Delta_i v_i^*(\mathbf{x}-\mathbf{e}_i)$ is increasing in x_0/r from Condition 3.c of $\tilde{\mathcal{V}}_i$, for $k \le i \le n-1$ (we can apply Condition 3.c of $\tilde{\mathcal{V}}_i$ since $x_0 > z_n^* - 1$ implies that $x_0 > z_i - 1$). Therefore $\Delta_{0n}\tilde{T}_k v$ is increasing in x_0/r for $x_0 \le z_k^*$.

Assume now that $z_{i-1}^* < x_0 \le z_i^*$ with $k < i \le n$. We consider this case if and only if $z_{i-1}^* < z_i^*$. From Condition 1 of $\tilde{\mathcal{V}}_{n-1}, \cdots, \tilde{\mathcal{V}}_i$, we have

$$\begin{aligned}
\Delta_0 v_{n-1}^*(\mathbf{x}-\mathbf{e}_0) &= \Delta_{n-1} v_{n-1}^*(\mathbf{x}-\mathbf{e}_{n-1}) + \Delta_{0n-1} v_{n-1}^*(\mathbf{x}-\mathbf{e}_0-\mathbf{e}_{n-1}) \\
&= \Delta_{n-1} v_{n-1}^*(\mathbf{x}-\mathbf{e}_{n-1}) + \Delta_0 v_{n-2}^*(\mathbf{x}-\mathbf{e}_0) \\
&= \cdots \qquad (41) \\
&= \sum_{j=i}^{n-1} \Delta_j v_j^*(\mathbf{x}-\mathbf{e}_j) + \Delta_0 v_{i-1}^*(\mathbf{x}-\mathbf{e}_0)
\end{aligned}$$

$[\mathbf{x}-\mathbf{e}_j]_0 > z_j^* - 1$ implies that $\Delta_j v_j^*(\mathbf{x}-\mathbf{e}_j)$ is increasing in x_0/r from 3.c of $\tilde{\mathcal{V}}_j$ and $[\mathbf{x}-\mathbf{e}_0]_0 > z_{i-1}^* - 1$ implies that $\Delta_0 v_{i-1}^*(\mathbf{x}-\mathbf{e}_0)$ is increasing in x_0/r from 3.d of $\tilde{\mathcal{V}}_{i-1}$.

For $x_0 > z_n^*$, $\Delta_{0n}\tilde{T}_k v(\mathbf{x})$ is increasing in x_0/r from Condition 3.a of $\tilde{\mathcal{V}}_n$. To conclude that $\Delta_{0n}\tilde{T}_k v(\mathbf{x})$ is increasing in x_0/r for $x_0 > z_n^* - 1$, we have to study now the limit points $x_0 = z_k^*, \cdots, z_n^*$. Let i be such that $k+1 \le i \le n-1$ and assume that $z_i^* < z_{i+1}^*$. Let us take $x_0 = z_i^*$, then

$$\begin{aligned}
&\Delta_{0n}\tilde{T}_k v(\mathbf{x}+\mathbf{e}_0/r) - \Delta_{0n}\tilde{T}_k v(\mathbf{x}) = \Delta_0 v_{n-1}^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_0) - \Delta_0 v_{n-1}^*(\mathbf{x}-\mathbf{e}_0) \\
&= \sum_{j=i+1}^{n-1} \Delta_j v_j^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_j) + \Delta_0 v_i^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_0) - \sum_{j=i}^{n-1} \Delta_j v_j^*(\mathbf{x}-\mathbf{e}_j) \\
&\quad - \Delta_0 v_{i-1}^*(\mathbf{x}-\mathbf{e}_0) \qquad (42) \\
&\ge \sum_{j=i+1}^{n-1} \Delta_j v_j^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_j) + \Delta_0 v_i^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_0) - \sum_{j=i}^{n-1} \Delta_j v_j^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_j) \\
&\quad - \Delta_0 v_{i-1}^*(\mathbf{x}-\mathbf{e}_0) \qquad (43) \\
&= -\Delta_i v_i^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_j) + \Delta_0 v_i^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_0) - \Delta_0 v_{i-1}^*(\mathbf{x}-\mathbf{e}_0) \\
&\ge -\Delta_i v_i^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_j) + \Delta_0 v_i^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_0) \qquad (44) \\
&= \Delta_{0i} v_i^*(\mathbf{w}-\mathbf{e}_i) \ge 0
\end{aligned}$$

(42) comes from (41). (43) comes from the monotonicity of $\Delta_j v_j^*(\mathbf{x}-\mathbf{e}_j)$ for $x_0 > z_j^* - 1 \geq z_i^* - 1$ (Condition 3.c of $\tilde{\mathcal{V}}_j$). (44) comes from the fact that $\Delta_0 v_{i-1}^*(\mathbf{x}-\mathbf{e}_0) \leq 0$ for $x_0 = z_i^*$.

Let's study the last limit point $x_0 = z_k^*$. Assume that $z_k^* = z_{k+1}^* = \cdots = z_{s-1}^* < z_s^*$. Then

$$\Delta_{0n}\tilde{T}_k v(\mathbf{x}+\mathbf{e}_0/r) - \Delta_{0n}\tilde{T}_k v(\mathbf{x}) = \Delta_0 v_{n-1}^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_0) - \Delta_k v_{n-1}^*(\mathbf{x}-\mathbf{e}_k)$$

$$= \sum_{j=s} \Delta_j v_j^*(\mathbf{x}+\mathbf{e}_0-\mathbf{e}_j) + \Delta_0 v_{s-1}^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_0) - \sum_{j=k}^{n-1} \Delta_j v_j^*(\mathbf{x}-\mathbf{e}_j) \tag{45}$$

$$\geq \sum_{j=s}^{n-1} \Delta_j v_j^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_j) + \Delta_0 v_{s-1}^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_0) - \sum_{j=k}^{n-1} \Delta_j v_j^*(\mathbf{x}+\mathbf{e}_0-\mathbf{e}_j) \tag{46}$$

$$= -\sum_{j=k}^{s-2} \Delta_j v_j^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_j) - \Delta_{s-1} v_{s-1}^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_{s-1}) + \Delta_0 v_{s-1}^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_0)$$

$$\geq -\Delta_{s-1} v_{s-1}^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_{s-1}) + \Delta_0 v_{s-1}^*(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_0) \tag{47}$$

$$= \Delta_{0s-1} v_{s-1}^*(\mathbf{w}-\mathbf{e}_{s-1}) \geq 0 \tag{48}$$

(45) comes from (40) and (41). (46) comes from 3.c of $\tilde{\mathcal{V}}_j$. (47) comes from the fact that $\Delta_j v_j^*(\mathbf{x}) \leq 0$ since $v_j^* \in \tilde{\mathcal{U}}_j$. (48) comes from Condition 2 of $\tilde{\mathcal{V}}_{s-1}$.

If $z_k^* = \cdots = z_n^*$, there is no difficulty since $\Delta_{0n}\tilde{T}_k v(\mathbf{x}) \geq 0$ for $x_0 > z_n^*$ and $\Delta_{0n}\tilde{T}_k v(\mathbf{x}) < 0$ for $x_0 \leq z_n^*$. We can use the same argument to the limit point $x_0 = z_n^*$ even if we don't have $z_k^* = \cdots = z_n^*$.

Let's show now that $\Delta_{0n}\tilde{T}_k v$ is decreasing in x_n. For $x_0 \leq z_n^*$, $\Delta_{0n}\tilde{T}_k v(\mathbf{x})$ doesn't depend on x_n from Condition 1 of $\tilde{\mathcal{V}}_n$ and for $x_0 > z_n^*$, $\Delta_{0n}\tilde{T}_k v(\mathbf{x})$ is decreasing in x_n from Condition 3.b of $\tilde{\mathcal{V}}_n$.

Let's show now that $\Delta_n\tilde{T}_k v$ is increasing in x_0/r.

$$\Delta_n\tilde{T}_k v(\mathbf{x}) = \begin{cases} \Delta_n v(\mathbf{x}-\mathbf{e}_k) & \text{if } x_0 \leq z_k^* \\ \Delta_n v(\mathbf{x}-\mathbf{e}_0) & \text{if } x_0 > z_k^* \end{cases}$$

First of all, for $0 \leq k \leq n$, $\Delta_n v(\mathbf{x}-\mathbf{e}_k) = \Delta_n v(\mathbf{x}-\mathbf{e}_n)$ for $x_0 \leq z_n^*$. Indeed

$$\Delta_n v(\mathbf{x}-\mathbf{e}_k) - \Delta_n v(\mathbf{x}-\mathbf{e}_n) = \Delta_{kn} v(\mathbf{x}-\mathbf{e}_k-\mathbf{e}_n) - \Delta_{kn} v(\mathbf{x}-\mathbf{e}_k) = 0$$

since $\Delta_{kn} v(\mathbf{x})$ doesn't depend on x_n for $[\mathbf{x}+\mathbf{e}_k]_0 \leq z_n^*$ from Condition 1 of $\tilde{\mathcal{V}}_n$. Therefore, when $x_0 \leq z_n^*$, $\Delta_n\tilde{T}_k v(\mathbf{x}) = \Delta_n v(\mathbf{x}-\mathbf{e}_n)$ is increasing in x_0/r from 3.c. Also, $\Delta_n\tilde{T}_k v(\mathbf{x})$ is increasing in x_0/r for $x_0 > z_n^*$ from Condition 3.c of $\tilde{\mathcal{V}}_n$. At the limit point $x_0 = z_n^*$,

$$\begin{aligned}\Delta_n \tilde{T}_k v(\mathbf{x}+\mathbf{e}_0/r) - \Delta_n \tilde{T}_k v(\mathbf{x}) &= \Delta_n v(\mathbf{x}-\mathbf{e}_0+\mathbf{e}_0/r) - \Delta_n v(\mathbf{x}-\mathbf{e}_n) \\ &\geq \Delta_n v(\mathbf{x}-\mathbf{e}_0+\mathbf{e}_0/r) - \Delta_n v(\mathbf{x}-\mathbf{e}_n+\mathbf{e}_0/r) \\ &= \Delta_{0n} v(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_0-\mathbf{e}_n) - \Delta_{0n} v(\mathbf{x}+\mathbf{e}_0/r-\mathbf{e}_0) \\ &\geq 0\end{aligned}$$

(Use 3.c and then 3.b to prove it).

In addition $\Delta_0 c$, $\Delta_n c$ and $\Delta_{0n} c$ doesn't depend neither on x_n, nor on x_0. Finally $\tilde{T}v$ verifies Condition 3 of $\tilde{\mathcal{V}}_n$ if $v \in \tilde{\mathcal{V}}_n$. Furthermore we can conclude that $v \in \tilde{\mathcal{V}}_n$ implies that $\tilde{T}v \in \tilde{\mathcal{V}}_n$ and by value iteration that $v^* \in \tilde{\mathcal{V}}_n$.

F Proof of Theorem 1

As we mentioned it earlier, $P(0)$ is true. Suppose that $P(n-1)$ is true. Lemma 1 states the first part of $P(n)$. From Lemma 2, using value iteration and the fact that the optimal infinite horizon policy can be obtained as the limit of finite horizon optimal policies, the second part of $P(n)$ is also true. As a result, for all n-class problems, the optimal policy is a WR policy.

Note furthermore that $(1-\lambda_n)v_{n-1}^*$ is the optimal value function corresponding to $\tilde{\mathbf{P}}_{n-1}(\mu, \boldsymbol{\lambda}^{n-1}, h+b_n, \mathbf{b}^{n-1} - b_i\mathbf{1}_{n-1}, r, \alpha)$. The optimal control only depends on the sign of $\Delta_{ij} v_{n-1}^*$ which is not affected by the factor $(1-\lambda_n)$ which shows the second part of the theorem.

G Heuristic Evaluation for the optimal rationing levels

For a k-class problem, if we consider the WR policy whose rationing work level vector is $(\mathbf{z}^k, z)$, the corresponding average cost $g(z)$ can be written as (see Gayon et al. 2006a):

$$g(z) = E\left[c_{k-1}(\mathbf{X}^{k-1})\right] - (b_k - b_{k+1})E\left[\sum_{i=1}^{k} X_i + \lfloor X_0 \rfloor\right] \tag{49}$$

We momentarily consider that X_0 and z_k are integers and we can then approximate $g(z)$ by

$$g(z) \simeq P\{X_0 \leq z_k\}g_{k-1} + \sum_{s=z_k+1}^{z}(h+b_k)sP\{X_0 = s\} - (b_k - b_{k+1})E\left[\sum_{i=1}^{k} X_i + X_0\right] \tag{50}$$

where $Y = z - \sum_{i=1}^{k} X_i - X_0$ is an $M/E_r/1$ queue-length process with arrival rate $\sum_{i=1}^{k} \lambda_i$. When $X_0 > z_k$, $Y = z - X_0$ because of the WR policy structure. No simple analytical expressions for the distribution of Y exists though, except for the exponential case ($r = 1$). We use then a geometric tail approximation for the queue length distribution of an M/G/1 queue (see Tijms 1994 , Karaesmen, Liberopoulos and Dallery 2003). Following this approach, it is then possible to approximate the value of z which minimizes $g(z)$.

Let us present briefly the approximation of the queue length distribution $\pi(n)$ of an $M/G/1$ queue described in detail in Tijms (1994). We denote by λ the arrival rate and by ρ the utilization rate. Let $f(t)$ be the probability density function of the processing time and L^* be its Laplace transform, then the approximation is given by

$$\pi(n) = \sigma\eta^n \quad \text{for } n \text{ sufficiently large}$$

where $\tau = 1/\eta$ is the smallest real solution strictly larger than 1 of the below equation

$$L^*(\lambda(1-\tau)) = \tau \tag{51}$$

In the case of an r-Erlang processing time with $1/\mu$ and utilization rate $\rho = \lambda/\mu$, (51) becomes

$$\left(\frac{r}{r+\rho(1-\tau)}\right)^r = \tau \tag{52}$$

In general, there is no closed form solution of the last equation but it's possible to obtain a numerical solution. Define the polynomial function f by:

$$f(x) = [r + \rho(1-x)]^r x - r^r$$

Then τ is the smallest real solution, strictly larger than 1, of the equation $f(x) = 0$. The derivative f' of f is then given by:

$$f'(x) = [r + \rho - \rho x)]^{r-1}[r + \rho - \rho(1+r)x]$$

and

$$f'(x) = 0 \Leftrightarrow x = \frac{r+\rho}{\rho(1+r)} \equiv x_1 \text{ or } x = \frac{r+\rho}{\rho} \equiv x_2$$

It is straightforward to show that f is decreasing on the interval $[y_1, y_2]$ and that $\tau \in [y_1, y_2]$. A dichotomy algorithm then gives a numerical expression of τ.

Tijms also proposes an expression for the constant σ that is asymptotically exact. In order to simplify the final form, we simply assume that the approximation given by equation (51) is valid for all $n > 1$ as in Jemai and Karaesmen (2003) and that $\pi(0) = 1 - \rho$ where ρ is the utilization rate. With the normalization condition, we obtain

$$\sigma = \frac{\rho}{\eta}(1-\eta)$$

In section **??**, Y is approximated by an $M/E_r/1$ queue-length process with utilization rate $\rho_k = \sum_{i=1}^{k} \lambda_i/\mu$. With the queue size distribution heuristic introduced above, we have

$$\begin{cases} P[Y=0] = 1-\rho_k \\ P[Y=j] = \frac{\rho_k}{\eta_k}(1-\eta_k)\eta_k^j \quad \text{if } j>0 \end{cases} \tag{53}$$

where η_k is the real solution strictly smaller than 1 of the following equation

$$\left(\frac{r}{r+\rho_k(1-1/\eta_k)}\right)^r = 1/\eta_k$$

In general, there is no closed form solution of the last equation but its numerical solution is straightforward. It is also possible to obtain an approximate value $\tilde{\eta_k}$ of η_k by taking $\tilde{\eta_k}$ such that the mean size of the approximate queue be equal to the exact mean size of the $M/G/1$ queue given by the Pollaczek-Khintchine formula

$$\frac{\rho_k}{1-\eta_k} = \rho_k + \frac{\rho_k^2(1+c_v^2)}{2(1-\rho_k)} \Rightarrow \tilde{\eta_k} = \rho_k \frac{2-\rho_k(1-c_v^2)}{2-\rho_k^2(1-c_v^2)}$$

where c_v is the coefficient of variation of the processing time. In the case of an r-Erlang processing time, $c_v^2 = 1/r$. We have tested but not reported the performance of the heuristic with an approximated $\tilde{\eta_k}$. It works well when r is small and tends to deteriorate when r is increasing.

Using (53), (50) becomes

$$\begin{aligned} g(z) &= P\{Y > z - z_k\} g_{k-1} + \sum_{s=z_k+1}^{z} (h+b_k) s P\{Y = z-s\} - (b_k - b_{k+1}) E\left[z - Y\right] \\ &= g_{k-1}\frac{\rho_k}{\eta_k}\eta_k^{z-z_k} + (h+b_k)\sum_{s=z_k+1}^{z-1} s\frac{\rho_k}{\eta_k}(1-\rho_k)\rho_k^{z-s} + (h+b_k)(1-\rho_k)z \\ &\quad + (b_k - b_{k+1})\left(\frac{\rho_k}{1-\eta_k} - z\right) \end{aligned} \tag{54}$$

We can then evaluate the difference $\Delta g(z) = g(z+1) - g(z)$

$$\Delta g(z) = -\frac{\rho_k}{\eta_k}\eta_k^{z-z_k}\left[(1-\eta_k)(g_{k-1} - (h+b_k)z_k) + \eta_k(h+b_k)\right] + h + b_{k+1}$$

which is nondecreasing in z. The cost $g(z)$ is convex and its minimum is attained at $\min\{z \in \mathrm{I\!R} | \Delta g(z) > 0\}$, that is at z where,

$$z = z_k + \frac{\ln \frac{\eta_k}{\rho_k} \frac{h+b_{k+1}}{\eta_k(h+b_k)+(1-\eta_k)(g_{k-1}-(h+b_k)z_k)}}{\ln \eta_k} = z_{k+1} \tag{55}$$

We do not round off z in order to keep track of the production information. Using the value of z_{k+1} and (54), a direct computation leads to the expression of g_k

$$\begin{aligned} g_k &= \frac{\rho_k}{\eta_k}\left[(\frac{\eta_k}{\rho_k}z_{k+1} - \frac{\eta_k}{1-\eta_k})(h+b_{k+1}) + \left(g_{k-1} - (z_k - \frac{\eta_k}{1-\eta_k})(h+b_k)\right)\eta_k^{z_{k+1}-z_k}\right] \\ &= (h+b_{k+1})\left(z_{k+1} + \frac{1-\rho_k}{1-\eta_k}\right) \end{aligned}$$

If we replace g_k by its value in equation (55), it gives

$$z_{k+1} = z_k + \log_{\eta_k} \frac{\eta_k(h + b_{k+1})}{\rho_k(h + b_k)\left[\eta_k + (1 - \eta_k)\frac{1-\rho_{k-1}}{1-\eta_{k-1}}\right]}$$

When we have obtained all the work rationing levels, we do

$$z_k = \frac{\lfloor r z_k + 1 \rfloor}{r}$$

in order to have $z_k \in \mathbb{N}_r$. We initialize the algorithm with $z_1 = 1 - 1/r$ and $\rho_1 = 1, \eta_0 = 0$.

Bios

Jean-Philippe Gayon is assistant professor in the Department of Industrial Engineering at INPG (Grenoble, France). He has a Ph.D. degree from ECP (Paris, France). His research interests are in stochastic modeling with applications to make-to-stock queues and dynamic pricing.

Francis de Véricourt is Associate Professor of Operation Management at the Fuqua School of Business at Duke University. Prior to joining the faculty of Duke University, he was a visiting researcher at MIT. His main research interest is in the optimization of supply chains and service delivery systems. He has published articles in leading academic and professional journals such as *Management Science*, *Operations Research*, and *IIE Transactions*. He is an Associate Editor for *Operations Research* and *IIE Transactions*. Francis is a native of Paris, France, where he graduated.

Fikri Karaesmen is Associate Professor in the Department of Industrial Engineering at Koç University (Istanbul, Turkey). He has a B.S. degree from METU (Ankara, Turkey) and a Ph.D. degree from Northeastern University (Boston, MA, USA). His research interests are in stochastic models of production and inventory systems and service systems. His papers have appeared in *IIE Transactions*, *Management Science*, *Manufacturing & Service Operations Management*, *Operations Research* and other journals. He is an Associate Editor for *4OR*, *IEEE Transactions on Automation Sciences and Engineering* and *IIE Transactions*.