A representation theorem for lattices via set-colored posets

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Abstract

This paper proposes a representation theory for any lattice via set-colored posets, in the spirit of Birkhoff for distributive lattices, and Korte and Lovász [1985], Edelman and Sacks [1988] for upper locally distributive lattices and convex geometries.

We show that set-colored posets captures the order induced by join-irreducible elements of a lattice as Birkhoff's representation does for distributive lattices, i.e. to study lattices theory is related to the coloring of the join-irreducible elements. We also survey some consequences of this representation on lattice theory and the lattice of Moore families.

Keywords: lattice, representation theory, upper locally distributive lattice, closure system, antimatroid, set-colored poset.

This paper is motivated by representation theory and its algorithmic consequences for combinatorial objects structured as lattices. Whenever you are familiar with Birkhoff's theorem, the intuition behind this new representation is the following: *Take a poset, say $P = (X, \leq)$, a set of colors M and color the elements of P by subsets of M. Then the set of colors of all ideals of P has a lattice structure and every lattice can be obtained in this way. For example if each element of P has only exactly one color then the obtained lattice is an antimatroid. Moreover, if any pair of elements have different colors, then the lattice is distributive.*

The question " *Why develop lattice theory?*" was considered by Birkhoff [5, 6] and extended by Wille [40, 12] using Formal Concept Analysis (FCA).

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The purpose of this paper is to present a new representation of lattices that allow us to understand lattices theory from an algorithmic point of view. Let us first recall the famous Birkhoff's representation theorem for distributive lattices. All results in this paper can be stated for the dual, by replacing join-irreducible with meet-irreducible elements.

Theorem 1 *[3] Any distributive lattice L is isomorphic to the lattice of all order ideals $\mathcal{I}(J(L))$, where $J(L)$ is the poset induced by the set of join-irreducible elements of L.*

Birkhoff's theorem has been widely used to derive algorithms in many areas. In fact, whenever a set of objects has a structure isomorphic to a distributive lattice, then there exists a poset where the set of its order ideals is isomorphic to the lattice, e.g. stable marriage [15], stable allocation [1], minimum cuts in a network [19], etc...

For general lattices, the unique well known representation is based on sets [4] or a binary relation between join-irreducible and meet-irreducible elements; called the bipartite irreducible poset $Bip(L) = (J(L), M(L), \not\leq_L)$ by Markowsky [28, 29, 30] or a context $B(L) = (J(L), M(L), \leq_L)$ by Wille in the framework of Formal Concept Analysis (FCA) [40, 12]. From the FCA perspective, elements in $J(L)$ are interpreted as objects and those of $M(L)$ are understood as properties or attributes characterizing these objets. Several other lattice representations have been proposed such as closure systems, implicational systems, join-core, but all of them use exponential size (see [10, 8, 22, 26]). Lattices representation, FCA and its algorithmic aspects are essential topics in data analysis, as they aim at identifying knowledges and restructuring them as a hierarchy (the reader is referred for examples to the ICFCA series of conferences). Over the past two decades, many algorithms have been introduced to consider the reconstruction of a lattice from its representation. Fortunately there exist linear time reconstruction algorithms for distributive lattice (see for example [17, 20, 39]). Enumeration or reconstruction algorithms for general lattices are equivalent to enumerate maximal bicliques of a bipartite graphs (see [13] for a detailed analysis).

Compared to the representation of distributive lattices, the bipartite irreducible poset does not take into account the order induced by join-irreducible elements and the fact that the elements of the lattice correspond to some order ideals of the poset $J(L) = (J(L), \leq)$. In this sense the proposed algorithms for the general case have a bad behavior whenever the lattice is distributive or near to be distributive.

In this paper, we propose a new representation for general lattices via set-colored posets which generalizes the notion of colored posets for representing upper locally

distributive lattice in [34, 35]. For a lattice L, this representation captures the order induced by join-irreducible elements $J(L)$ as Birkhoff's representation does for distributive lattices. First, we argue that this representation by the fact that the elements of a lattice L correspond to some order ideals of $J(L)$. When restricted to distributive lattices, we obtain an isomorphism between L and the order ideals of $J(L)$. Dilworth [9] has introduced upper locally distributive lattices and has observed that they are close to distributive lattices. Using set-colored posets we confirm this idea and for upper locally distributive lattices, we obtain a characterization strongly linked to that of distributive lattices, which can be also deduced from Korte and Lovász [25] and Edelman and Sacks [11] works. Recently, Knauer [24, 23] has confirmed this observation using antichains partition. Moreover Magnien *et al.* [27] have shown that configurations of a Chip-Firing game are structured as upper locally distributive lattice (see Kolja[24]).

The rest of the paper is structured as follows: in Section 1, we introduce the set-colored posets and the lattice of ideal color sets. We also characterize set-colored posets which are associated to a lattice. In Section 2, we list some applications of set-colored posets and the last section is devoted to some algorithmic consequences of our representation Theorem.

1 Set-Colored Posets

In this section we first introduce the notion of set-colored posets and some notations that will be used throughout this paper. For definitions on lattices and ordered sets not given here, see [8, 38, 32].

A partial order (or poset) on a set X is a binary relation $\leq$ on X which is reflexive, anti-symmetric and transitive, denoted by $P = (X, \leq)$. A set $I \subseteq X$ is said to be an ideal if $x \in I$ and $y \leq x$ implies $y \in I$. For an element $x \in X$ we associate the unique ideal $\downarrow x = \{y \in X \mid y \leq x\}$. The set of all ideals of P is denoted by $\mathcal{I}(P)$.

Let $A \subseteq X$, an element $z \in X$ is an upper bound of A if $x \leq z$ for any $x \in A$. If z is said to be the least upper bound if $z \leq z'$ for all upper bounds z' of A. Dually, we define the greatest lower bound. A partial order $L = (X, \leq)$ is called a lattice if for every two elements $x, y \in X$ both the least upper bound and the greatest lower bound exist, denoted by $x \vee y$ and $x \wedge y$. Let $L = (X, \leq)$ be a lattice. The element $z \in X$ is a join-irreducible (resp. meet-irreducible) if $z = x \vee y$ (resp. $z = x \wedge y$) implies $z = x$ or $z = y$. The set of all join-irreducible (resp. meet-irreducible) elements of L is denoted by $J(L)$ (resp. $M(L)$).

Let L be a finite lattice and $x, y \in L$. We will use the following arrow relations [12], that are weakening of the so called perspectivities relations defined in lattices (see [14]) : $x \swarrow y$ means that x is a minimal element of $\{z \in L \mid z \not\leq y\}$, $x \searrow y$ means that y is a maximal element of $\{z \in L \mid z \not\geq x\}$ and $x \updownarrow y$ means that $x \swarrow y$ and $x \searrow y$. Recall that $\swarrow, \searrow, \updownarrow$ are relations defined on $J(L) \times M(L)$.

A set-coloring γ for a poset $P = (X, \leq)$ is a function that assigns a set of colors to every element in X such that for all $x, y \in X$, the sets of colors $\gamma(x)$ and $\gamma(y)$ are disjoint whenever $x < y$. In other words, γ is a set-coloring of the comparability graph of P, as introduced in [37].

Definition 1 *A set-colored poset, denoted by $P = (X, \leq, \gamma, M)$, is the poset $(X, \leq)$ equipped with a set coloring $\gamma : X \to 2^M$ where M is a set of colors. A set-colored poset is said to be a* ***proper colored poset*** *if the color set of any element is a singleton, i.e., the coloring γ is a function from X to M.*

Figures 1(a) and 1(b) show two examples of set-colored posets.

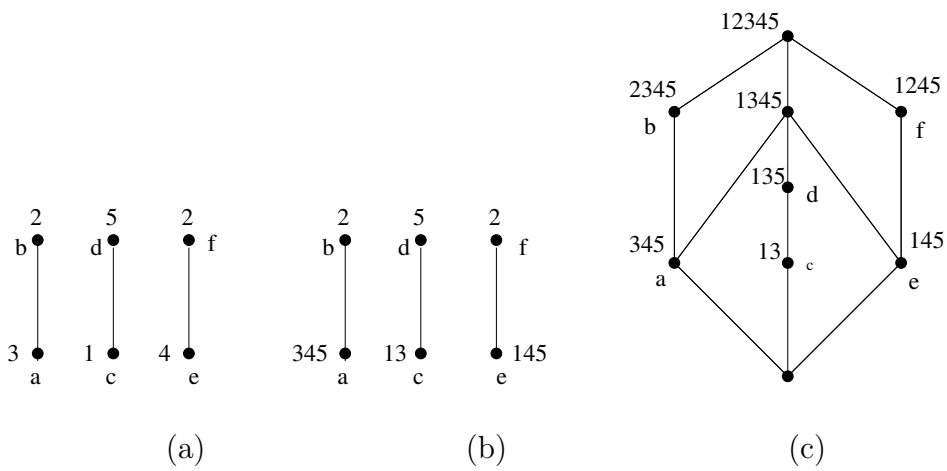


Figure 1: (a) a colored poset, (b) a set-colored poset and (c) the lattice of ideal color sets of the set-colored poset in (b).

Let $P = (X, \leq, \gamma, M)$ be a set-colored poset. A subset $C \subseteq M$ is said to be an *ideal color set* if there exists an ideal I of P such that $C = \gamma(I) = \bigcup_{x \in I} \gamma(x)$. In Figure 1(b), $C = \{1, 3, 4, 5\}$ is an ideal color set, since $C = \gamma(\{a, c\})$. Note that two different ideals of P can have the same color set, i.e. if $\gamma(\{c, e\}) = \gamma(\{a, c\}) = \{1, 3, 4, 5\}$.

The set of all ideal color sets of P, denoted by $\mathcal{C}(P)$ has a lattice structure as shown in the following:

Proposition 1 *Let $P = (X, <, \gamma, M)$ be a set-colored poset. Then $\mathcal{C}(P)$ ordered under set-inclusion is a lattice.*

Proof: It suffices to show that $\mathcal{C}(P)$ is a closed under union and containing the empty set. First we show that $\mathcal{C}(P)$ is closed under union. Let C_1, C_2 be two ideal colors sets of P. Then there exist two ideals I_1 and I_2 such that $\gamma(I_1) = C_1$ and $\gamma(I_2) = C_2$. Since ideals are closed under union, thus $I_1 \cup I_2$ is an ideal and therefore $C_1 \cup C_2$ are its colors. Moreover the ideal color set corresponding to the empty order ideal is empty. $\square$

Figure 1(c) shows the lattice of the ideal colors sets of the set-colored poset in Figure 1(b).

Proposition 2 *Let $P = (X, <, \gamma, M)$ be a set-colored poset. The mapping $gen : \mathcal{C}(P) \to \mathcal{I}(P)$ defined by $gen(C) = \{x \in X \mid \gamma(\downarrow x) \subseteq C\}$ is an order embedding. Moreover, $gen(C)$ is the unique largest ideal I of P with $\gamma(I) = C$.*

Proof: Let C_1, C_2 be two ideal colors sets of P. We show that $C_1 \subseteq C_2$ iff $gen(C_1) \subseteq gen(C_2)$. First suppose $C_1 \subseteq C_2$ and let $x \in gen(C_1)$. Then $\gamma(\downarrow x) \subseteq C_1 \subseteq C_2$ which implies that $x \in gen(C_2)$. Now suppose that $C_1 \not\subseteq C_2$. Then there exists $c \in C_1 \setminus C_2$ and $x \in X$ such that $c \in \gamma(x)$, which implies that $\gamma(x) \not\subseteq C_2$ and therefore $gen(C_1) \not\subseteq gen(C_2)$ since $x \notin gen(C_2)$.

Assume that there exist two different maximal (under inclusion) ideals I and J with $\gamma(I) = \gamma(J) = C$. Then $\gamma(I \cup J) = C$ since ideals are closed under union, and thus contradicts the fact that I and J are maximal under inclusion with $\gamma(I) = \gamma(J) = C$. $\square$

Let us now examine the consequences of these definitions.

2 Representing a lattice by a set-colored poset

In this section we show that any lattice L can be represented by a set-colored poset P_L such that its associated lattice $\mathcal{C}(P_L)$ is isomorphic to L.

Definition 2 *Let $L = (X, \leq)$ be a lattice. We denote $P_L = (J(L), \leq, \gamma, M(L))$, the set-colored poset defined by the following set-coloring :*

$$\gamma : J(L) \to 2^{M(L)}, \text{ with } \gamma(j) = \{m \in M(L) \mid j \swarrow m\}$$

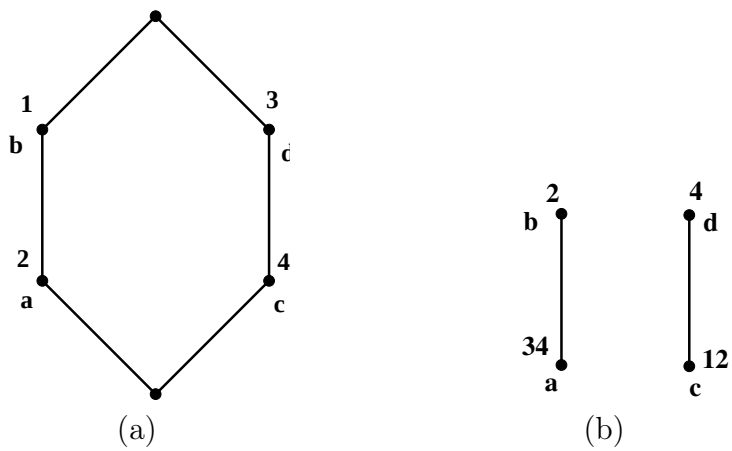


Figure 2: (a) a lattice L in which join-irreducible (resp. meet-irreducible) elements are labelled with letters (resp. numbers) and (b) its associated set-colored poset P_L.

Lemma 1 *Let $L = (X, \leq)$ be a lattice and the mapping $\varphi : L \to \mathcal{C}(P_L)$ with $\varphi(a) = \gamma(J(a))$, where $J(a) = \{j \in J(L) \ \mid \ j \leq a\}$, then:*

1. *For every $a \in X$, $\varphi(a) = \{m \in M(L) \ \mid \ a \not\leq m\}$*
2. *For every $a, b \in X$, $a \leq b$ iff $\varphi(a) \subseteq \varphi(b)$, i.e. φ is an order embedding.*

Proof:

1. Let $m \in M(L)$ and $a \not\leq m$. Then there exists $j \leq a$ such that $j \swarrow m$. Thus $m \in \gamma(j)$ which implies $m \in \gamma(J(a)) = \varphi(a)$, since $j \in J(a)$.

   Now let $m \in \varphi(a)$. Then there exists $j \in J(a)$ such that $j \swarrow m$. This means that $j \not\leq m$ and then $a \not\leq m$ since $j \leq a$.

2. Let $a \leq b$, for $a, b \in L$. Then $a \not\leq m$ implies $b \not\leq m$. So $\varphi(a) \subseteq \varphi(b)$.

   Now let $\varphi(a) \subseteq \varphi(b)$. This means, that for all $m \in M(L)$, $a \not\leq m$ implies $b \not\leq m$. Suppose $a \not\leq b$. Then there exists $m \in M(L)$ with $a \not\leq m$ et $b \leq m$, which is a contradiction.

□

We can now formulate our main representation theorem :

Theorem 2 *Any lattice L is isomorphic to the lattice of colored ideals of its poset P_L.*

Proof: To this aim, let us prove that for a lattice $L = (X, \leq)$, $\varphi : L \to \mathcal{C}(P_L)$ is an order-isomorphism, with $\varphi(a) = \gamma(J(a))$ and $\varphi^{-1}(C) = \bigvee gen(C)$.

Using Lemma 1, the mapping φ is an order embedding and therefore one-to-one. It remains to show that φ is onto.

Let $C \in \mathcal{C}(P_L)$ and $a = \bigvee gen(C)$. Then $a \in L$ since the supremum always exists for a finite lattice. It suffices to show that $\varphi(a) = C$.

Let $m \in C$. Then there exists $j \leq a$ such that $m \in \gamma(j)$, i.e. $j \not\leq m$. Thus $a \not\leq m$ and $m \in \varphi(a)$.

Conversely, let $m \in \varphi(a)$. Then there exists $j \leq a$ such that $j \not\leq m$. Suppose $m \notin C$. This implies that $j' \leq m$ for all $j' \in gen(C)$. Therefore $a = \bigvee gen(C) \leq m$ which contradicts $a \not\leq m$ and then $m \in \varphi(a)$. $\square$

As Birkhoff's Theorem 1 which provides for distributive lattices not only a compact representation via a poset but also some structural insights that can be used algorithmically [17], our Theorem 2 does the same for arbitrary lattices. This representation is compact since $|L|$ can be exponential in $|P_L|$.

Before discussing some of the consequences of this result, let us first characterize set-colored posets which are isomorphic to the set-colored poset P_L for some lattice L. Figure 3 shows three set-coloring of the same poset and none of them is isomorphic to some P_L for a lattice L.

Recall the characterization of those bipartite posets which are isomorphic to $Bip(L) = (J(L), M(L), \not\leq)$ for some lattice L.

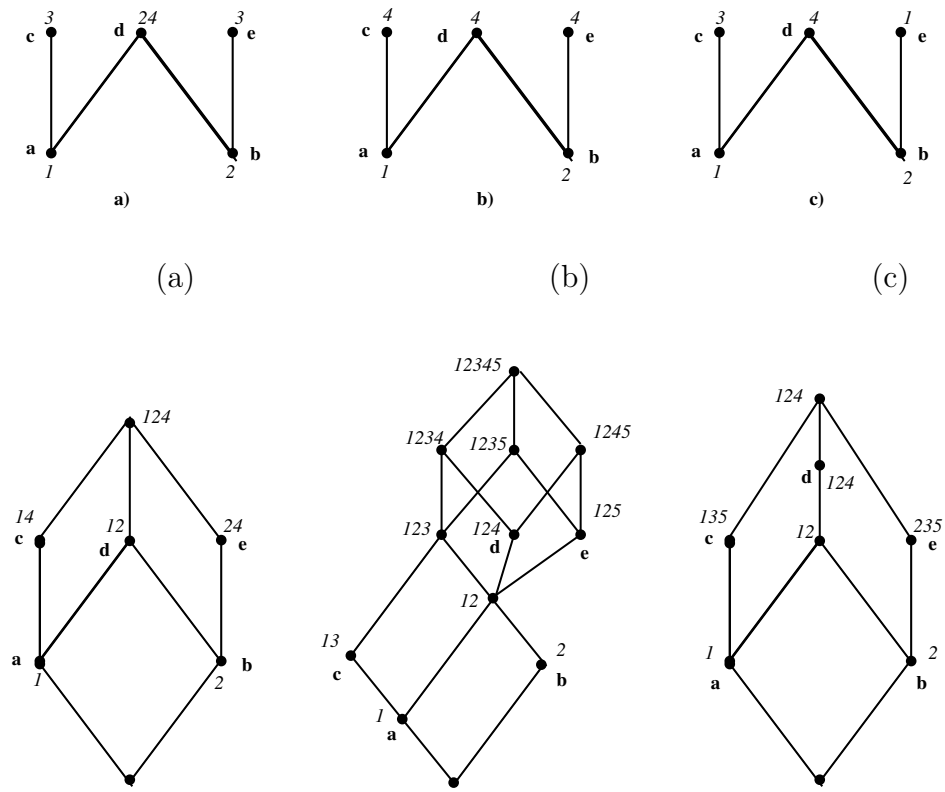
Theorem 3 *[30] Let $B = (X, Y, E)$ be a bipartite poset. B is isomorphic to $Bip(L) = (J(L), M(L), \not\leq)$ for some lattice L if and only if the following holds:*

1. *For all $x \in X$, if $W \subseteq X$ is such that $N(x) = N(W)$, then $x \in W$.*
2. *For all $y \in Y$, if $V \subseteq Y$ is such that $N(y) = N(V)$, then $y \in V$.*

where $N(x)$ is the neighboorhood of x in B.

Using the characterization of Theorem 3, we derive a characterization of set-colored posets which are isomorphic to P_L for some lattice L.

Theorem 4 *Let $Q = (X, \leq, \gamma, M)$ be a set-colored poset. Then Q is isomorphic to P_L for some lattice L iff the following conditions are satisfied :*



(a) (b) (c)

(d) (e) (f)

Figure 3: (d), (e) and (f) are respectively the lattices of ideal color sets of set-colored posets in (a),(b) and (c). In (a) the element d is not a join-irreducible, in (b) the element a is not comparable to e and in (c) the number of colors is greater than the number of meet-irreducible elements.

1. *For all $A \subseteq X$ and $x \in X$, $\gamma(\downarrow x) = \gamma(\downarrow A)$ implies $x \in A$ (***irreducible condition***), where $\downarrow x = \{y \in X \mid y \leq x\}$ and $\gamma(\downarrow A) = \bigcup_{a \in \downarrow A} \gamma(a)$.*

2. *For all $x, y \in X$, $\gamma(\downarrow x) \subseteq \gamma(\downarrow y)$ implies $x \leq y$ (***ordering condition***).*

3. *For all $m, n, p \in M$, $\beta(m) \neq \beta(n)$ or $\beta(m) = \beta(n) \cap \beta(p)$ implies $m = n$ or $m = p$, where $\beta(m) = \{a \in X \mid m \notin \gamma(\downarrow a)\}$.*

Proof: These conditions are obviously necessary, let us examine their sufficiency. Using irreducible condition and Lemma 1, there is a bijection between X and $J(L)$, and by (c) $(X \leq)$ is isomorphic to $(J(L), \leq)$. Similarly there is a bijection between

Y and $M(L)$, that associate to $m \in Y$ the ideal color set $C = \gamma(\{a \mid m \notin \gamma(\downarrow a)\})$. Now, let $m \in \gamma(a)$, $a \in X$. Then $\gamma(\downarrow a) \not\subseteq C$ since $m \notin C$. $\square$

Clearly conditions of Theorem 4 can be checked in polynomial time in the size of the colored poset and thus it can be recognized in polynomial time wether a given set-colored poset is isomorphic to some P_L for a lattice L.

3 Applications on particular classes of lattices

We show in this section how Theorem 2 unifies many results of lattice theory. First, we derive the famous Birkhoff's representation for distributive lattices Theorem 1, then we consider Korte and Lovász's results [25] or Edelman and Sacks's results [11] for upper locally distributive lattices. We also show that Theorem 2 yields a characterization of the lattice of all Moore families and also of extremal and semidistributive lattices.

3.1 Distributive lattices

Distributive lattices have used either theoretically or algorithmically in several areas. In fact several combinatorial objects can be structured as a distributive lattices (see for example, Knauer [24] who gives a list of problems from graphs).

Theorem 5 *Let L be a lattice. Then the following are equivalents:*

1. *L is distributive*
2. *L is isomorphic to the lattice of all ideals of the poset induced by join-irreducible elements of L*
3. *For each $j \in J(L)$ there exists a unique $m \in M(L)$ such that $j \swarrow m$ and dually, for each $m \in M(L)$ there exists a unique $j \in J(L)$ such that $j \swarrow m$.*
4. *$P_L = (J(L), \leq, \gamma, M)$ is a proper colored poset and γ is bijective.*

Proof:

The equivalence between 1. and 2. is due to Birkhoff's Theorem 1. The equivalence between 1. and 3. is due to a Theorem of Wille [12]. To show that 2. is equivalent to 4., it suffices to note that there is a bijection between order ideals of $(J(L), \leq)$ and the ideal color sets of P_L. Indeed, two different ideals have different colors since γ is bijective. The equivalence between 3. and 4. is by definition of P_L.

$\square$

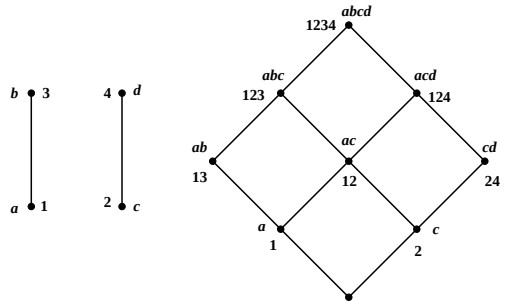


Figure 4: The distributive case

3.2 Upper locally distributive lattices

Dilworth [9] has introduced upper locally distributive lattices and has observed that they were close to distributive lattices. Upper locally distributive lattices have been rediscovered many times, and have several names in the literature (such as join-distributive lattices [11], or antimatroids [25], ...). For a survey, see Monjardet [31] which contains many characterizations. Here we will consider an upper locally distributive lattice as an antimatroid. Based on our work [34], Knauer [24] has given an equivalent characterization and gives a list of applications of upper locally distributive lattices. Using the characterization of upper locally distributive lattice by arrows relations, we obtain the following:

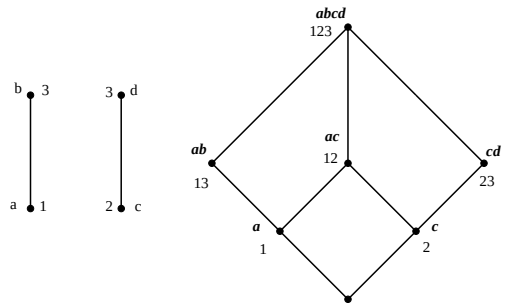


Figure 5: An example of locally distributive lattice and its associated proper colored poset

Theorem 6 *[34] For a lattice L, the following statements are equivalent:*

1. *L is upper locally distributive.*

2. *For each $j \in J(L)$ there exists a unique $m \in M(L)$ such that $j \swarrow m$.*

3. *$P_L = (J(L), \leq, \gamma, M(L))$ is a proper colored poset.*

Proof: The equivalence between 1. and 2. is due to Ganter and Wille [12] (dual of Theorem 44). The equivalence between 3. and 4. is by definition of P_L. $\square$

Therefore, distributive and upper locally distributive lattices can be represented by proper colored posets. Furthermore, Theorem 6 confirms Dilworth's observation, since it differs only slightly from Theorem 5.

3.3 The lattice of Moore families

In this section we recall the characterization of the lattice of Moore families using proper colored poset [16].

Let X be an n-set and 2^X its power set. A Moore family on X is a family of subsets of X closed under set-intersection and containing the set X. A Moore family is also known as a closure system; i.e. the set of all closed sets of a closure operator. The set $\mathcal{M}_n$ of all possible Moore families on an n-set X, ordered by set-inclusion is a lower locally distributive lattice. By Theorem 6 there exists a colored poset P_n such that the lattice $\mathcal{M}_n$ is dually isomorphic to the set of its ideal color sets.

Let Q be a boolean lattice on n atoms, say $a_0, a_1, ..., a_{n-1}$. We consider the mapping $\gamma : Q \to [0, 2^n - 1]$ as follows :

$$\gamma(x) = \begin{cases} 0 & \text{if x is the bottom element} \\ 2^i & \text{if } x = a_i \text{ for some } i \in [0, n-1] \\ \sum_{a \in J(x)} \gamma(a) & \text{otherwise} \end{cases} \qquad \text{(eq 1)}$$

where $J(x)$ is the set of all atoms below x in Q.

The application γ is a properly coloring since each element of 2^X has only one color. Moreover $x <_Q y$ implies $J(x) \subset J(y)$ and therefore $\gamma(x) \neq \gamma(y)$.

The poset P_n is defined as the disjoint sum of all intervals $[a, \top]$ of Q where a is an atom of Q and $\top$ the top element of Q, i.e. $P_n = \bigcup_0^{n-1} Q_i$ where Q_i is the induced poset by $[a_i, \top]$ in Q and the coloring is inherited from the coloring of Q (see Figure 6).

Proposition 3 *[16] There is a bijection between the ideal colors sets of P_n and the set of Moore families on an n-set.*

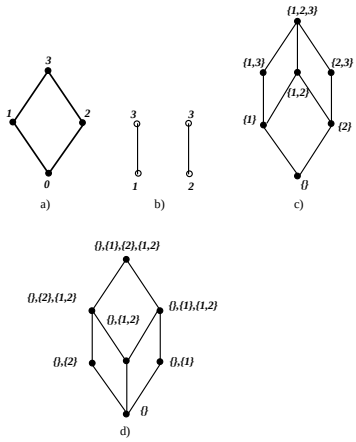


Figure 6: a) 2^2 the boolean lattice having two atoms, b) P_2 the properly colored poset, c) the ideal color sets lattice of P_2, and d) the lattice of Moore families on the set $X = \{1, 2\}$.

By applying the algorithm to generate ideal color sets for a colored poset, we found that the number of Moore families on 6 elements is exactly **75.973.751.474** (see [16] for more details). Recently, Colomb *et al.* [7] have obtained the number of Moore families on $n = 7$ using recursive decomposition of $\mathcal{M}_n$.

Now we derive an explicit characterization the lattice of all Moore families using the binary relation $R_n = (J_n, M_n, \leq)$ where J_n and M_n are respectively the set of join-irreducible and meet-irreducible elements. This is a rewriting of Proposition 3 using binary relation instead of colored poset. Let $J_n = \{a_{ij}$ such that $i \in [0, n-1], j \in [0, 2^{n-1} - 1]\}$, $M_n = \{k \in [1, 2^n - 1]\}$ and $a_{ij} \leq k$ iff $(2^i\ OR\ 2^j)\ AND\ k = k$ where OR and AND are the logical binary operations.

Corollary 1 *There is a bijection between maximal antichains (bicliques) of $R_n = (J_n, M_n, \leq)$ ($\bar{R}_n = (J_n, M_n, \not\leq)$) and the set of all Moore families on a n-set.*

3.4 Semidistributivity and extremality

In [18] meet-simplicial lattices were introduced. This class of lattices generalizes many known classes of lattices such as meet-extremal and meet-semidistributive lattices. Using set-colored posets the authors in [18] have characterized meet-semidistributive lattices as Nation [33] did for semidistributive lattices.

A lattice L is said to be meet-semidistributive if for all elements $x, y, z \in L, x \wedge y = x \wedge z$ implies $x \wedge y = x \wedge (y \vee z)$. A meet-semidistributive lattice is said semidistributive if for all elements $x, y, z, x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$. L is called meet-extremal if for all $x \in L$ $h(L) =\mid M(L) \mid$ where $h(L)$ is the size of a maximal chain in L. A meet-extremal lattice L is called extremal if $h(L) = J(L)$.

Let L be a lattice with n join-irreducibles and $\sigma = j_1, j_2, \ldots, j_n$ be an ordering of $J(L)$. We denote by Δ_i the colors that not appear in the i-1 first join-irreducibles in P_L, ie. $\Delta_i = |\gamma(j_i) \setminus \bigcup_{h=1}^{i-1} \gamma(j_h)|$. L is called meet-simplicial if there is a total ordering $\sigma = (j_1, ..., j_{|J(L)|})$ such that $\Delta_i \leq 1$.

The following results on extremal and meet-extremal theorems are written in terms of set-colored posets [30].

Theorem 7 *[18] A lattice L is semidistributive (resp. extremal) then there exists an ordering (resp. a linear extension) $\sigma = j_1, j_2, \ldots, j_n$ of $J(L)$ such that $|\Delta_i| = 1$.*

Theorem 8 *[18] If L is meet-extremal (resp. meet-semidistributive) lattice then there exists an ordering (resp. a linear extension) $\sigma = j_1, j_2, \ldots, j_n$ of $J(L)$ such that $|\Delta_i| \leq 1$.*

We conclude that meet-extremal and meet-semidistributive distributive lattices are meet-simplicial.

4 Discussion and opens problems

1. In order to formalize the proximity to distributive lattices, let us define a new lattice invariant called chromatic index : For any latttice L and its associated colored poset $P = (X, <, \gamma, M)$, we define $\chi(L)$ as $max_{x \in X}|\gamma(x)|$. We have noticed that for upper locally distributive lattices the chromatic index is one. A natural problem arises: **Find a characterization of lattices having chromatic index k, for every $k \geq 2$?**

   Recently, Beaudou et al [2] have shown that computing a minimal implicational basis from a set-colored poset and the opposite can be done in polynomial time whenever the chromatic index is constant.

2. Can we list the set of all ideal colors sets in $O(n^2)$ per element using polynomial space, where n is the number of vertices of $J(L)$? The best known complexity is related to the matrices product, i.e. $O(n^{2.38})$. Nourine and Raynaud [36] have given an $O(n^2)$ algorithm but using exponential space.

3. Can we compute for a lattice given by its set-colored poset, a minimal implicational basis in quasi-polynomial? The unique quasi-polynomial time algorithm for lattices corresponds to lattices isomorphic to an independence system (or hypergraph), where minimal basis is the set of all minimal transversal of the hypergraph. Recently, Kante *et. al* [21] have shown that the enumeration of minimal transversal of an hypergraph is polynomially equivalent to the enumeration of minimal dominating sets of a graph. Connected minimal dominating sets can be candidate for the general case?

We hope that this representation of lattices using set-colored posets could be helpful for studying lattice theory and its algorithmic aspects using the fact that set-colored posets is a simple generalization of Birkhoff's representation for distributive lattices.

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