Stéphan is 50!

Lucas Pastor

Old and new results on graph coloring

Sylvain Gravier, T. Karthick, Frédéric Maffray

Choosability in claw-free graphs

Definition

A graph G is k-***choosable*** if it is L-colorable for every assignment L that satisfies $|L(v)| = k$.

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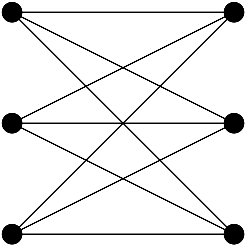

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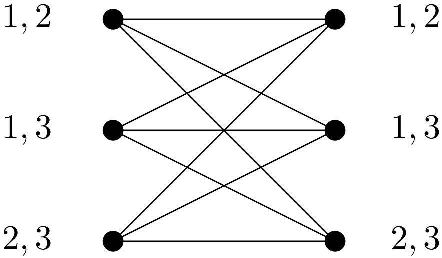

Conjecture *Vizing, Gupta, Albertson, . . .*

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Every claw-free graph G satisfies $\chi_\ell(G) = \chi(G)$.

One of the most important result:

Theorem *Galvin 1995*

Every line-graph G of a bipartite multigraph satisfies $\chi_\ell(G) = \chi(G)$.

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Theorem ***Gravier, Maffray, P. 2016***

Every claw-free perfect graph G with $\omega(G) \leq 4$ satisfies $\chi_\ell(G) = \chi(G)$.

Techniques

Theorem *Chvátal and Sbihi 1988*

Every claw-free perfect graph either has a clique cutset or is a peculiar graph or an elementary graph.

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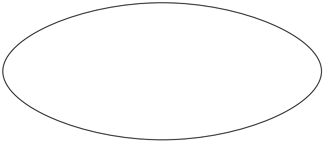
Lemma

Let G be a peculiar graph with $\omega(G) \leq 4$. Then G is 4-choosable.

G



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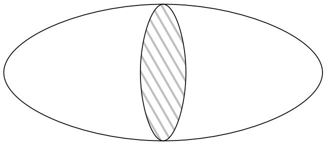

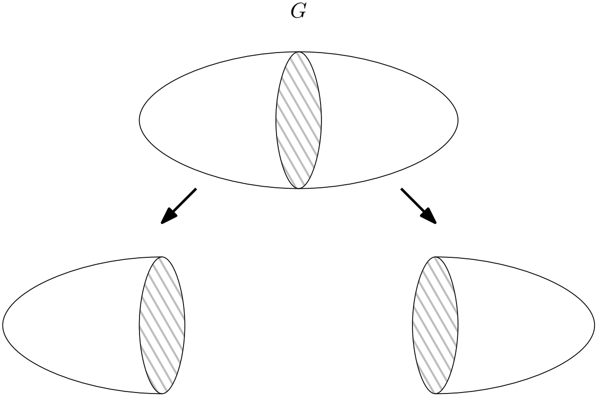
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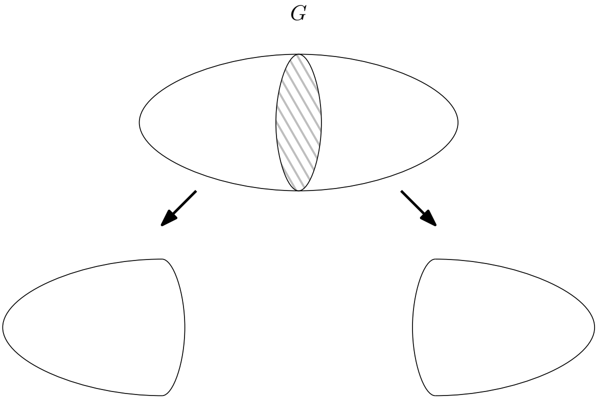
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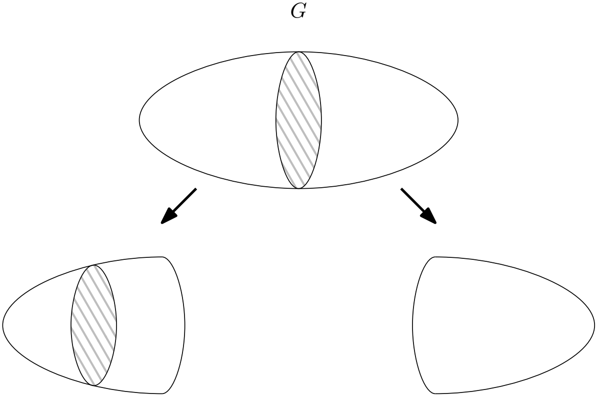

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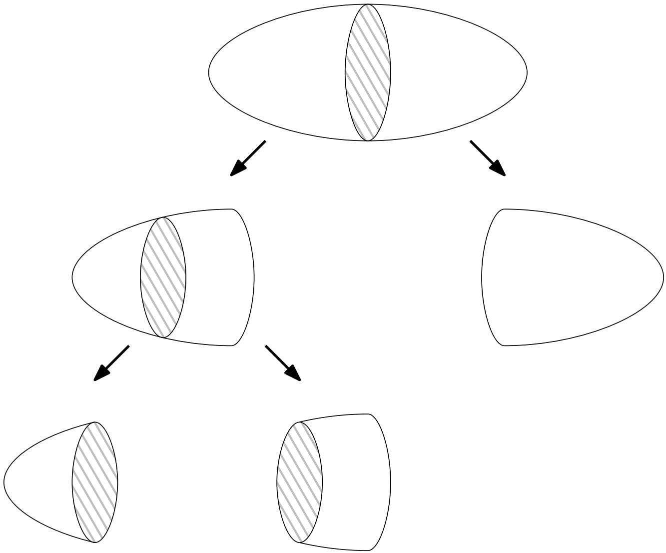
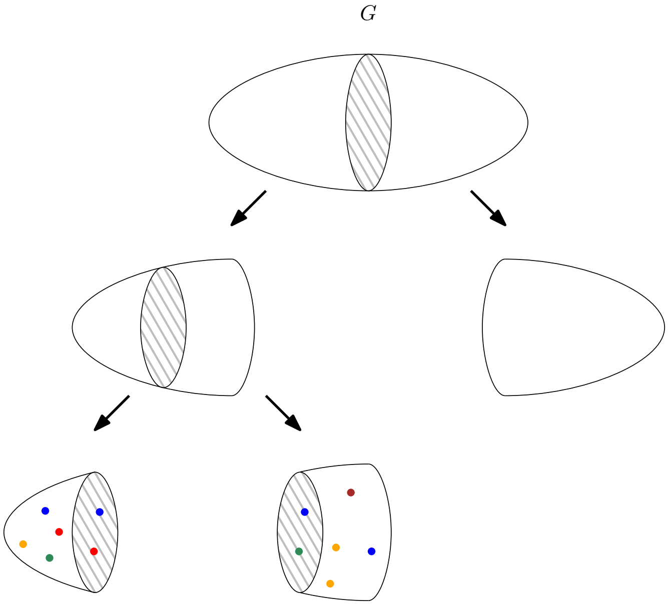
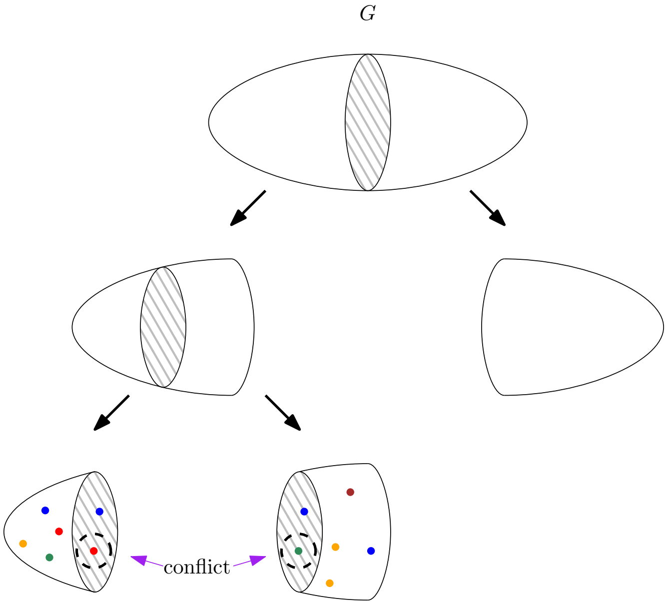




Color these graphs with any list assignment of size k and glue them back.



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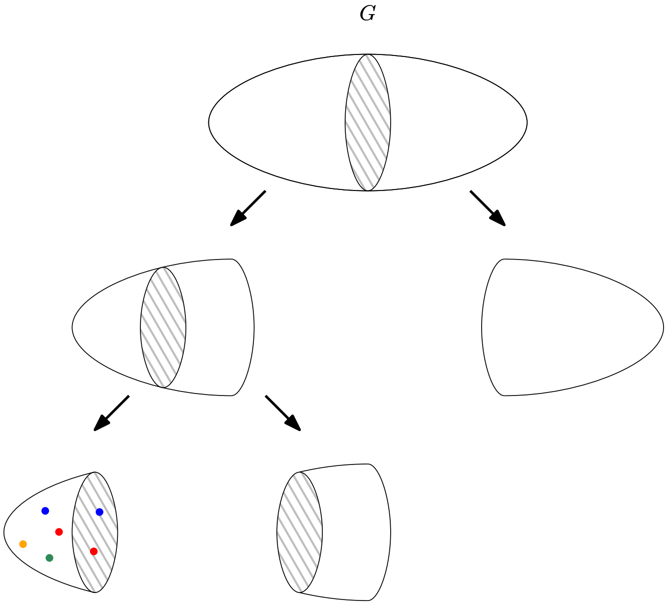
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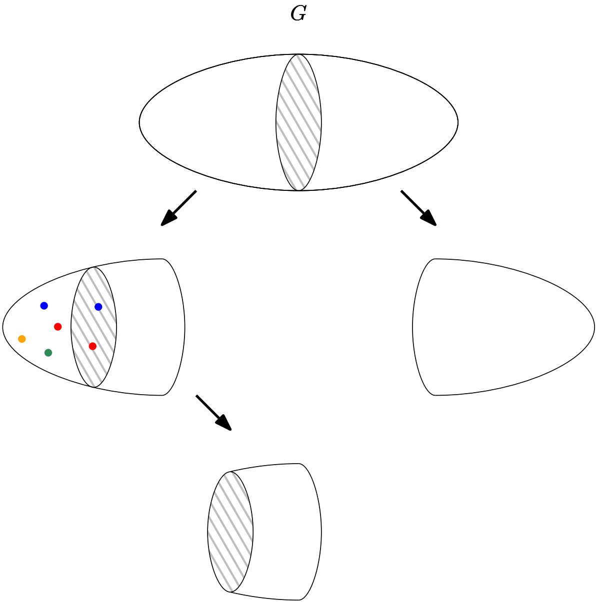
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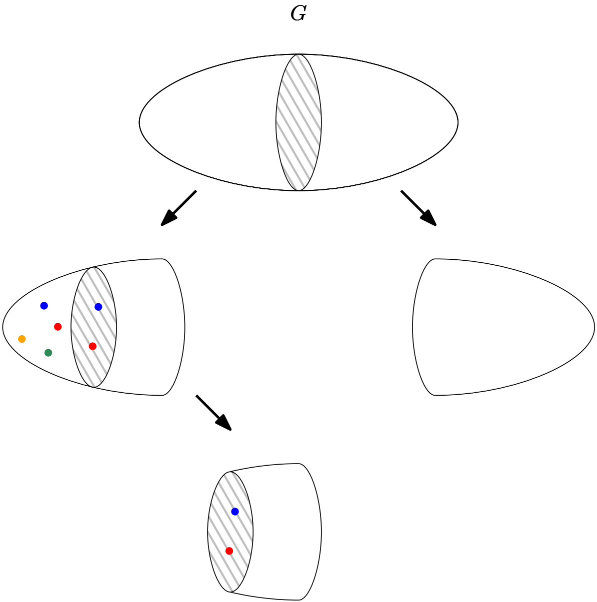
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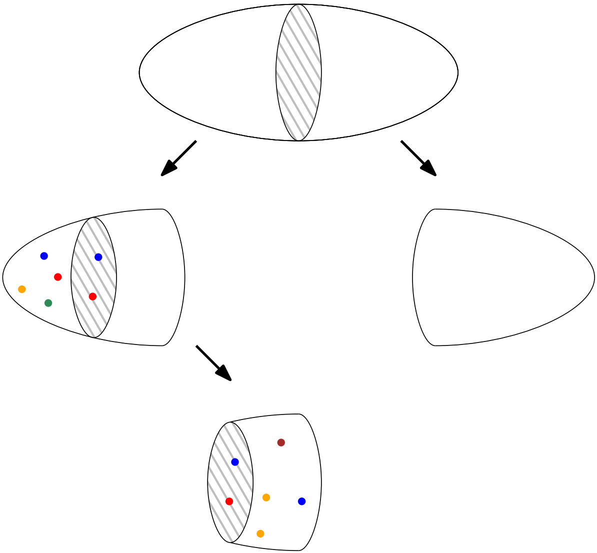

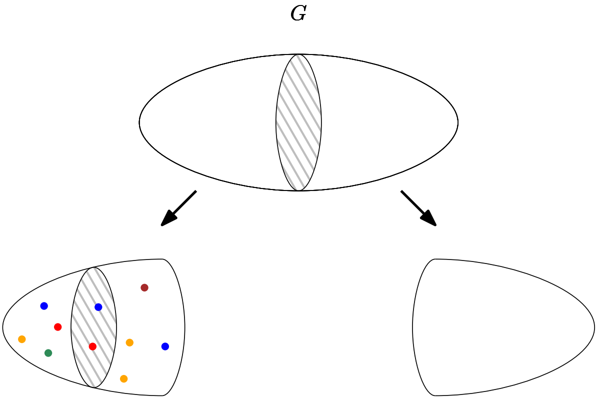
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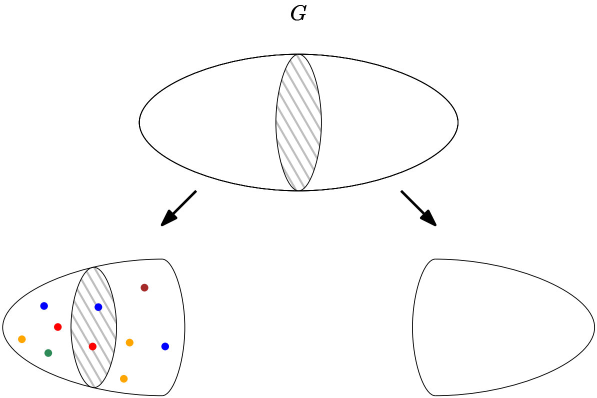
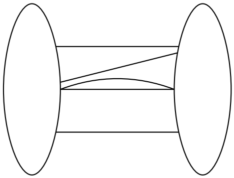


Each time, pick an *extremal* clique cutset.

How elementary graphs are built?

B



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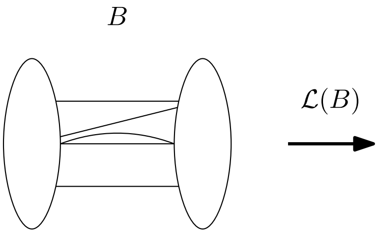

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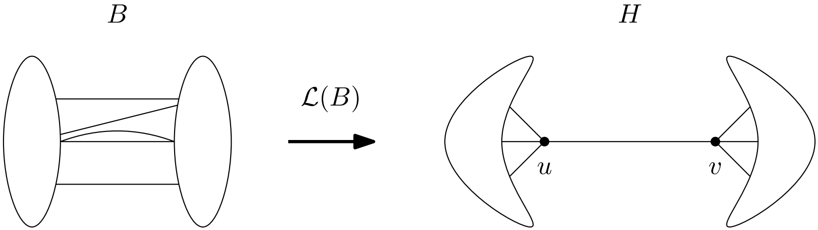

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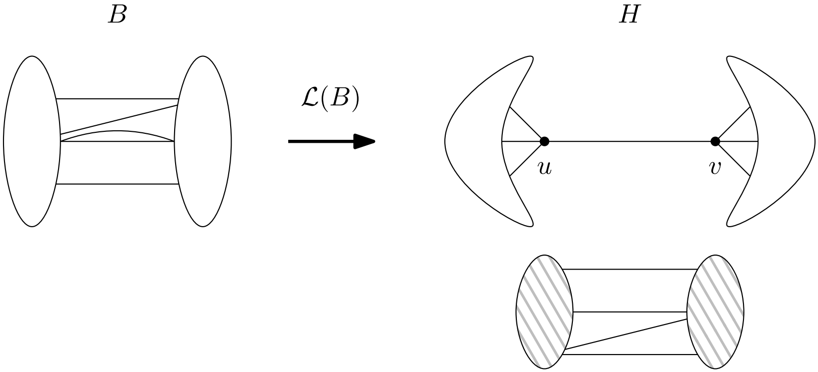

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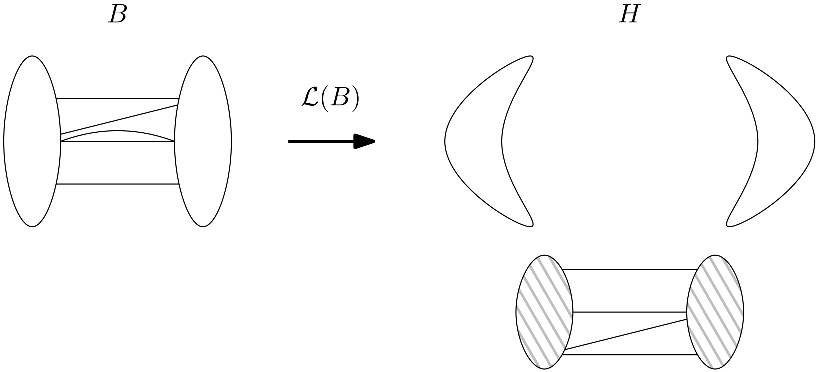

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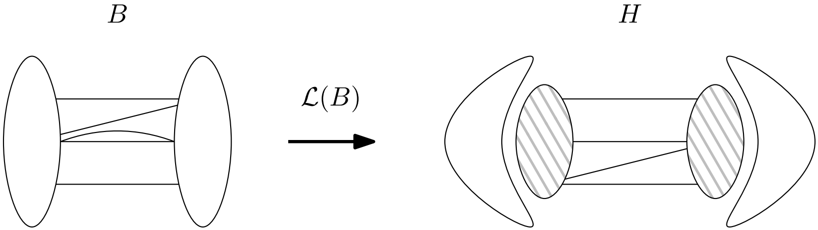

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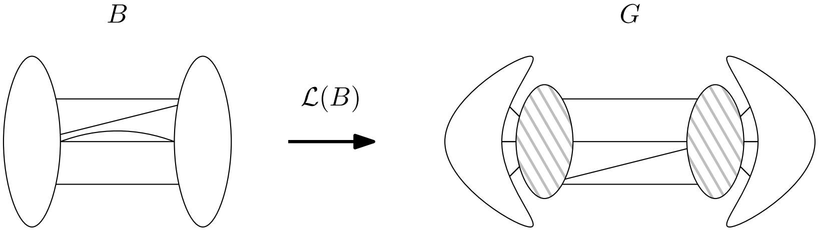

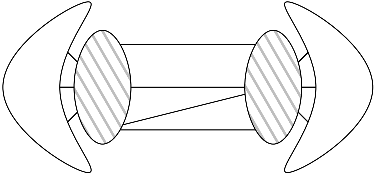

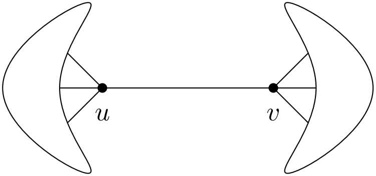

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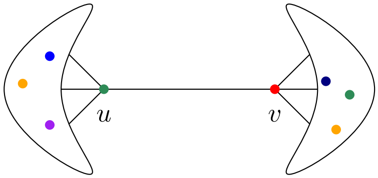
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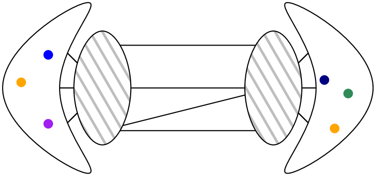

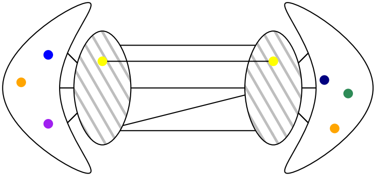

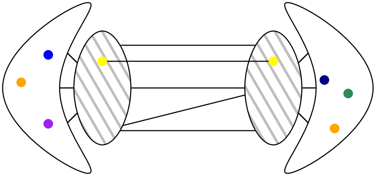

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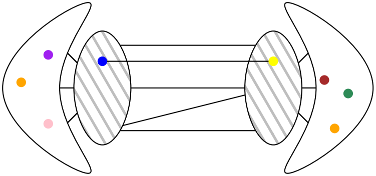

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Structure theorem of Chudnovsky and Plumettaz might give another point of view and new ideas.

Vertex coloring problem

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Still open for:

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2. (P_5, H)-free graphs where $H \in \{\overline{K_3 + O_2}, K_{2,3}, \text{dart}, \text{banner}, \text{bull}, \overline{2P_2 + P_1}\}$

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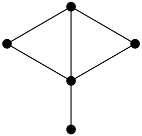
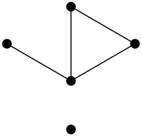
In total, this is 7 cases. We solve four of them by using a structural approach:

1. (P_5, dart)-free graphs
2. (P_5, banner)-free graphs
3. (P_5, bull)-free graphs
4. (fork, bull)-free graphs.

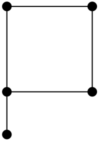
P_5



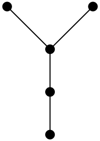
dart



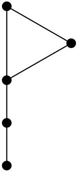
co-dart



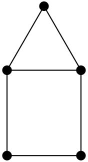
banner



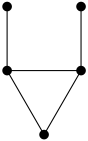
fork



hammer



house



bull

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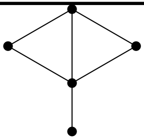

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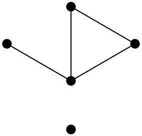

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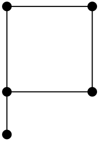

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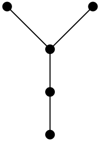

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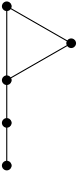

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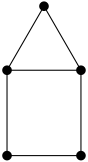

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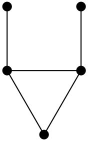

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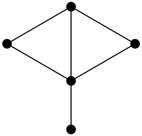

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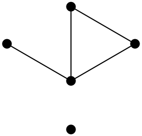

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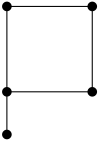

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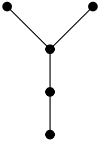

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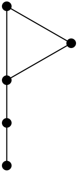

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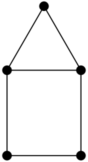

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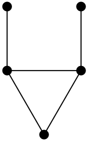

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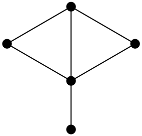

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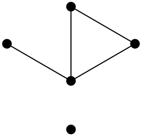

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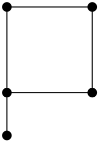

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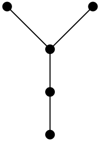

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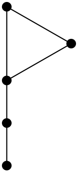

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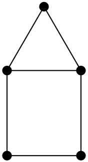

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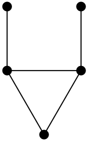

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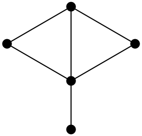
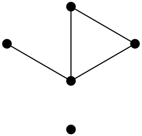
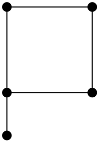


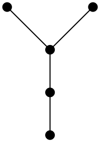
dart



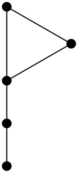
co-dart



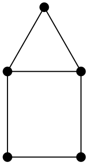
banner



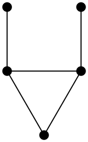
fork



hammer



house



bull

Definition

A class of graphs $\mathcal{G}$ is ***hereditary*** if it is closed under induced subgraphs.

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Corollary

The WVC problem can be solved in polynomial time in the class of (P_5, banner)-free graphs.

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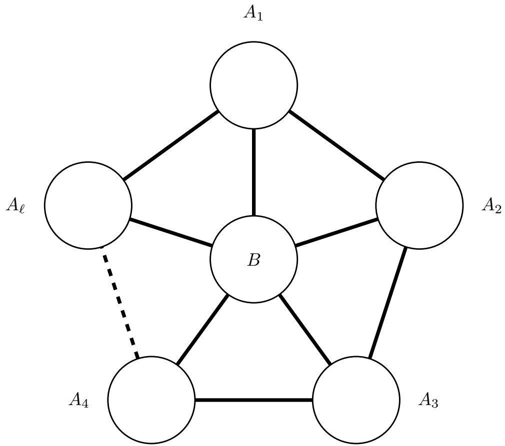
A_1
A_ℓ
A_2
B
A_4
A_3

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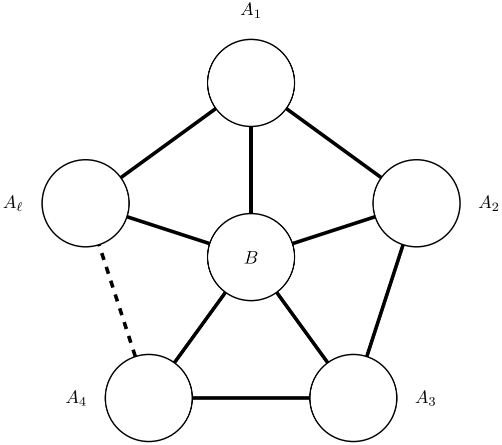

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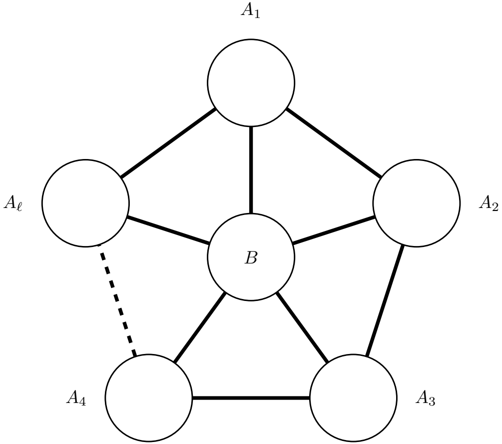

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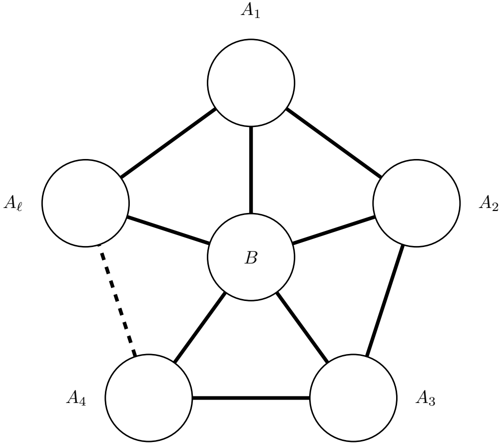

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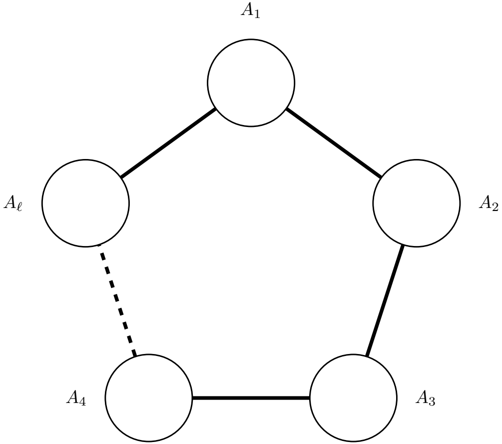

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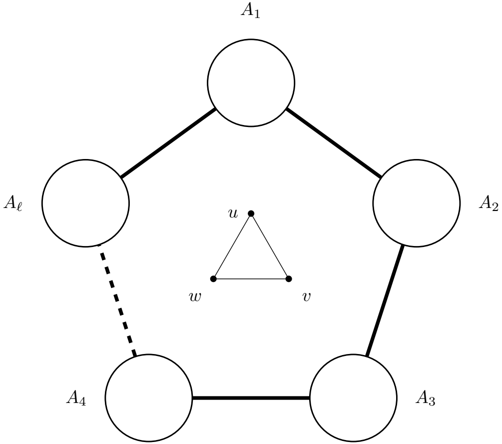

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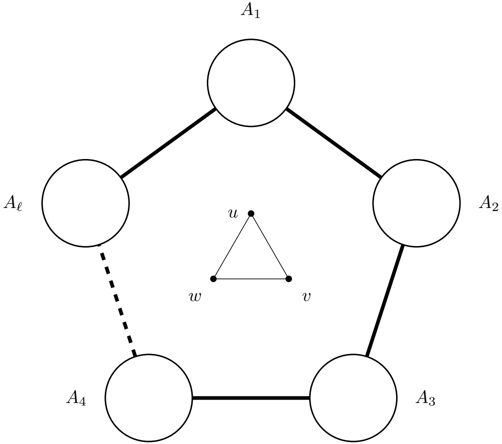

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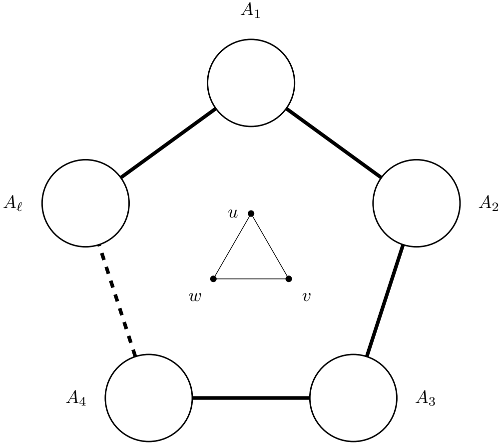

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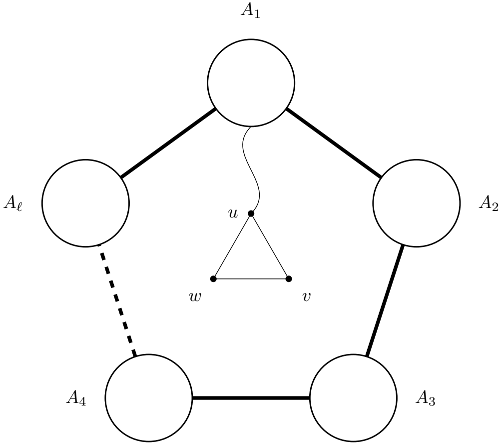

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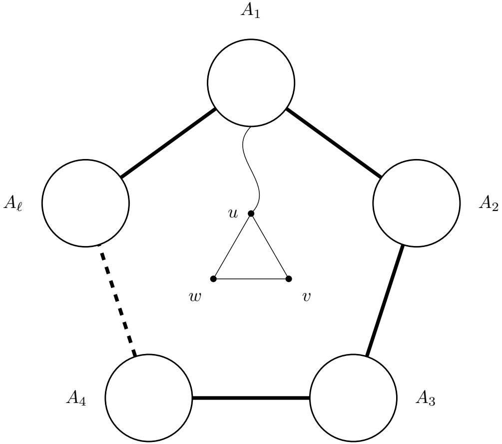

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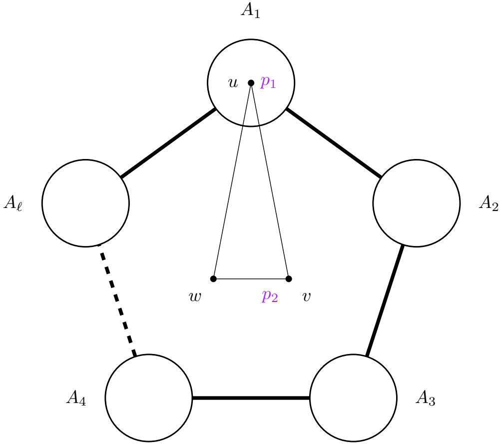

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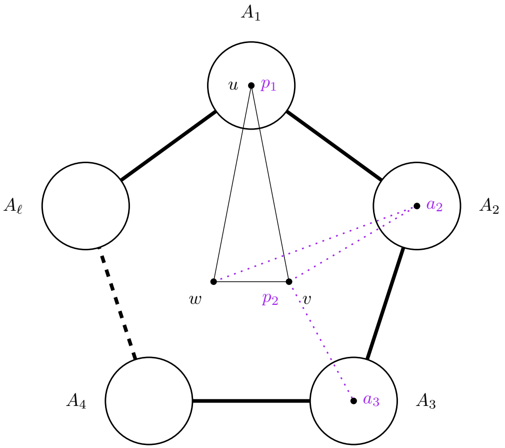

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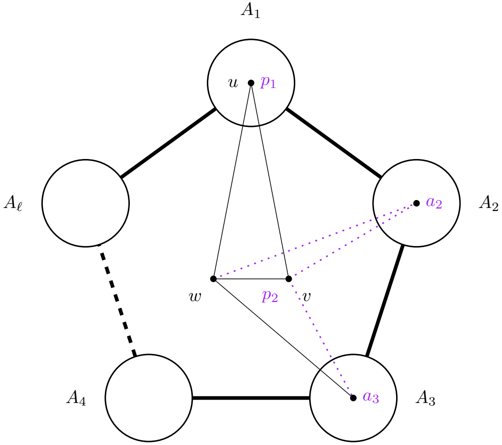

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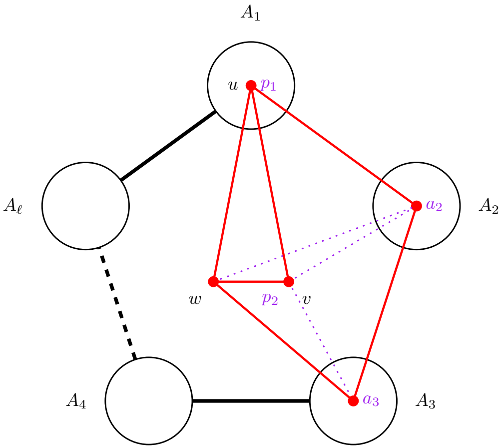

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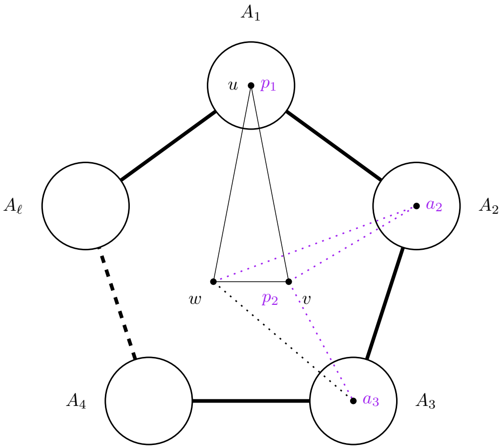

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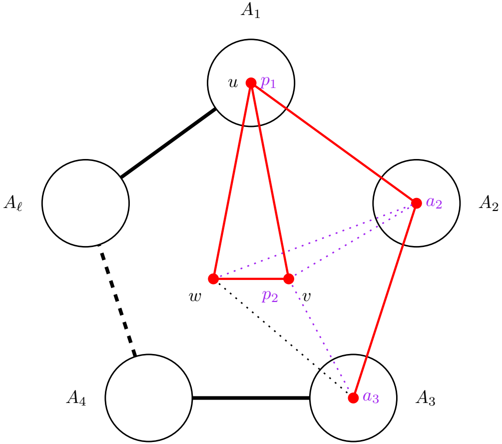

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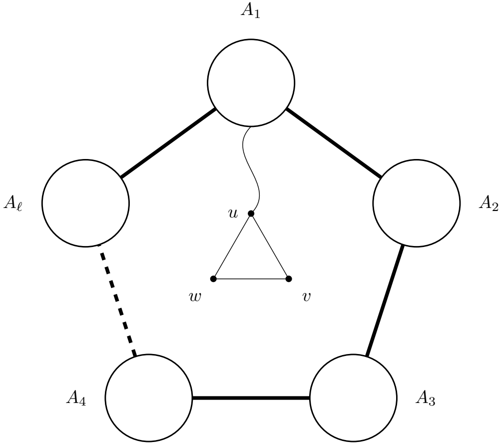

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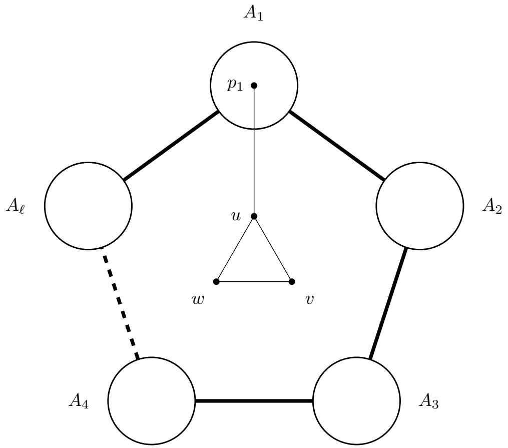

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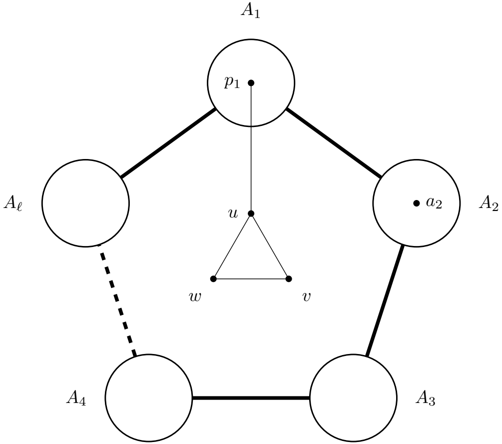

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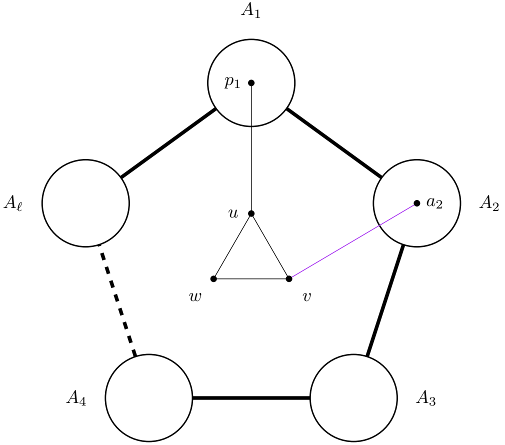

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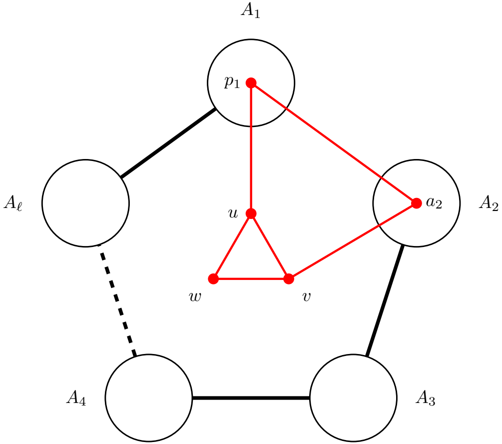

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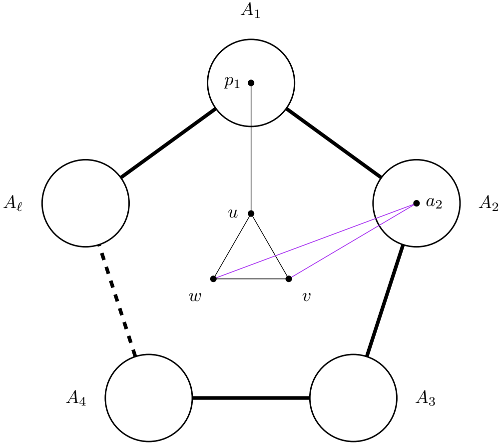

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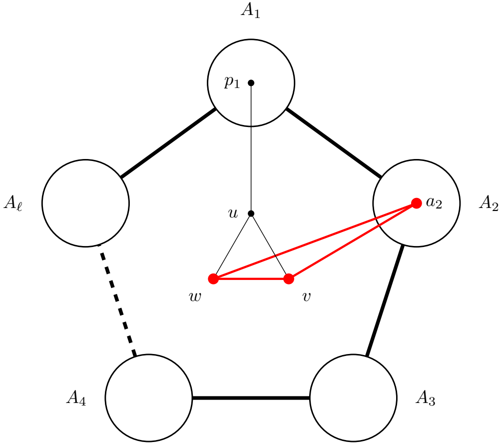

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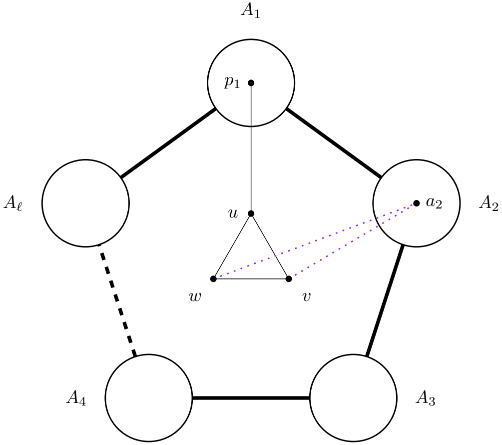

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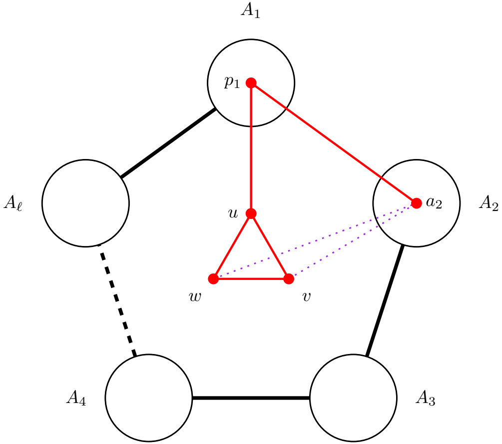

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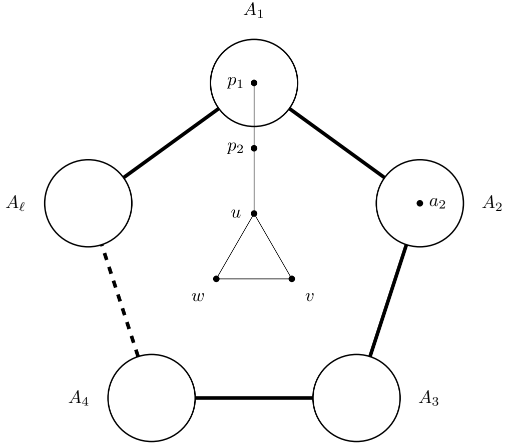

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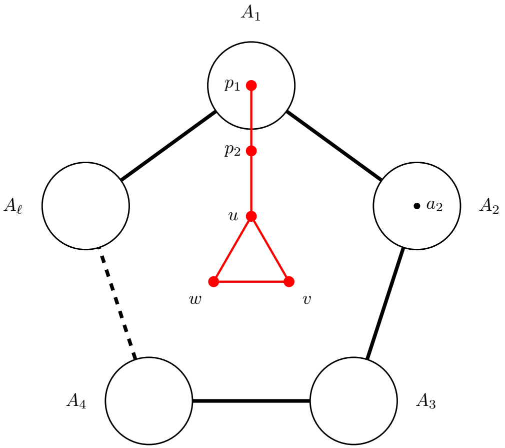

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Corollary

The WVC problem can be solved in polynomial time in the class of (P_5, bull)-free graphs.

Still open:

1. $(P_5, \overline{2P_2 + P_1})$-free
2. $(P_5, K_{2,3})$-free
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Conjecture

The WVC problem is polynomial time solvable for the last open cases.

Thank you for your attention.