A PhD thesis I would like to do again.

Lucas Pastor

UNIVERSITÉ Clermont Auvergne

LIMOS

List-coloring claw-free perfect graphs

Definition

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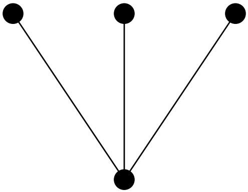
Definition

A graph G is ***claw-free*** if it does not contain any induced subgraph isomorphic to $K_{1,3}$ (the claw).



claw

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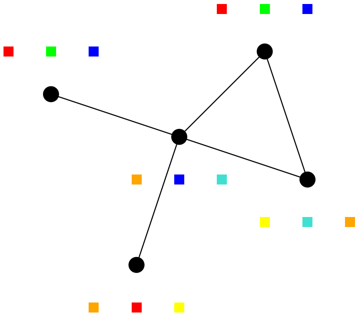

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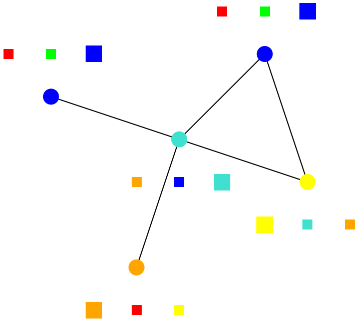

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Given an integer k, a graph G is k-***choosable*** if it L-colorable for every assignment L that satisfies $|L(v)| = k$ for all $v \in V(G)$.

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Conjecture *Vizing 1975*

For every graph G, $\chi_\ell(\mathcal{L}(G)) = \chi(\mathcal{L}(G))$.

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Theorem

Let G be a claw-free perfect graph $\omega(G) \leq 4$. Then $\chi_\ell(G) = \chi(G)$.

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We achieved that thanks to a decomposition theorem of Chátal and Sbihi ***and*** a structural theorem of Maffray and Reed.

List-coloring claw-free perfect graphs

Theorem ***Chvátal, Sbihi 1988***

Every claw-free perfect graph either has a ***clique-cutset***, or is a ***peculiar graph***, or is an ***elementary graph***.

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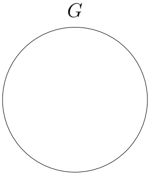

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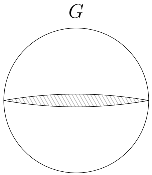

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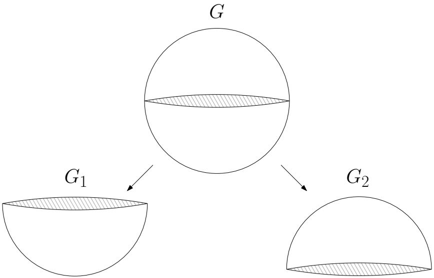
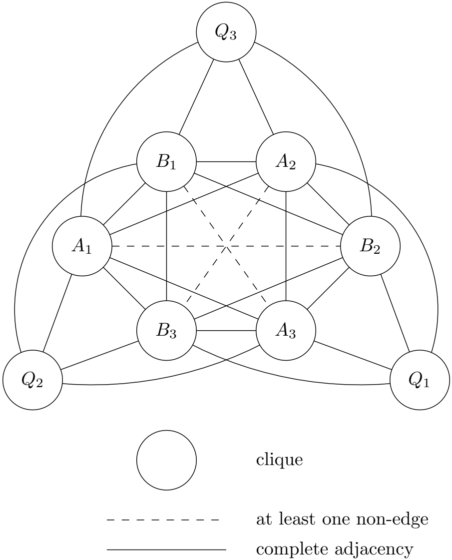


Peculiar graphs



Lemma

Let G be a peculiar graph with $\omega(G) \leq 4$ (***unique*** in this case). Then, $\chi_\ell(G) = \chi(G)$.

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Proof

- If some pair of non-adjacent vertices u, v share a color, let $\alpha = c(u) = c(v)$ and we can easily color $G - \{u, v\}$ without using α.
- If no such pair exists, we can find a coloring by Hall's theorem (we have enough color to directly color G).

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A graph G is elementary if and only if it is an augmentation of a the line-graph H of a bipartite multigraph B.

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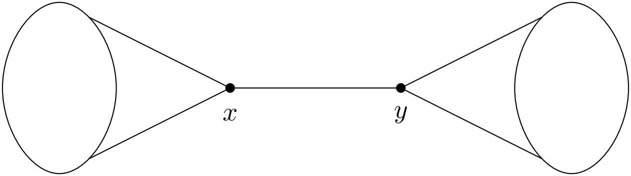

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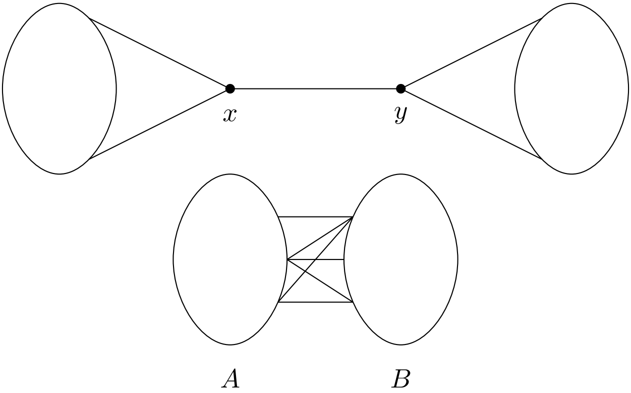

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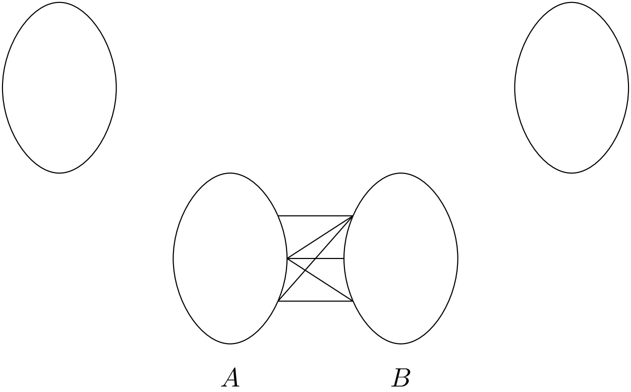

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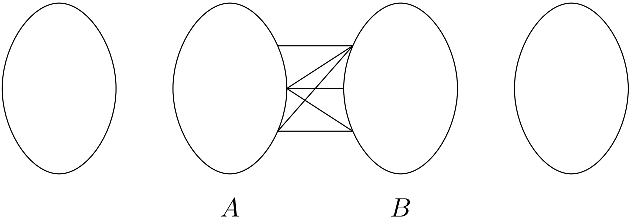

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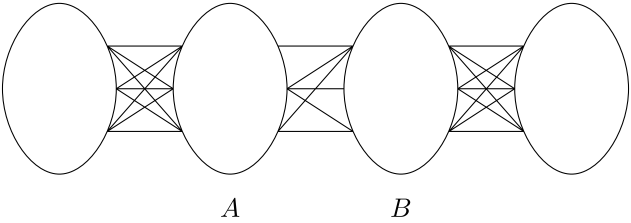

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By induction on h, the number of augmented flat edges :

- If $h = 0$, use Gavlin's theorem (LCC is true for line-graphs of bipartite multigraphs).
- If $h > 0$, we use a gadget.

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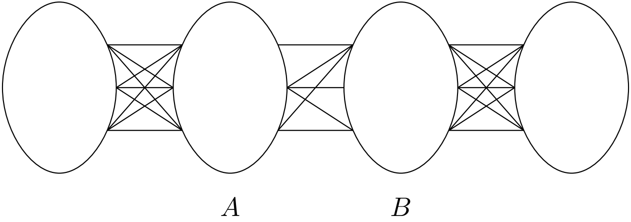

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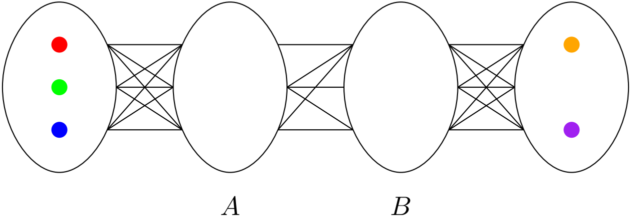

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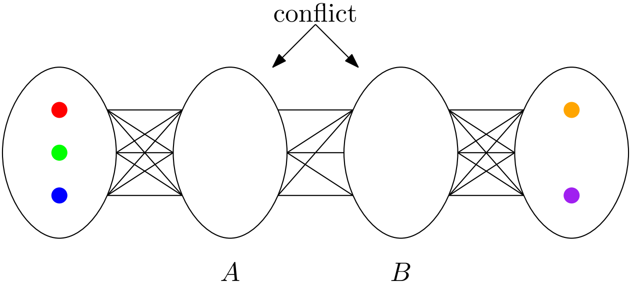

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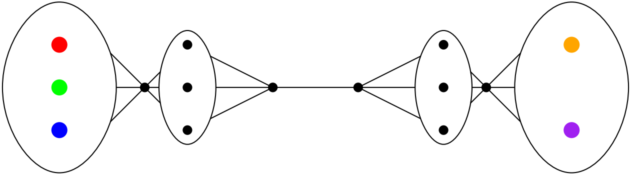

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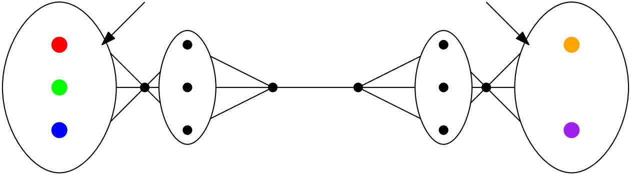

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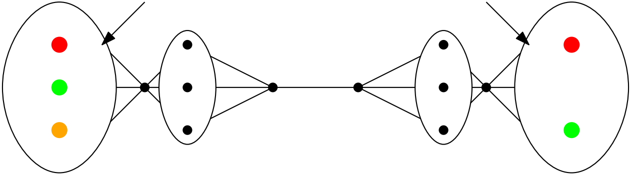

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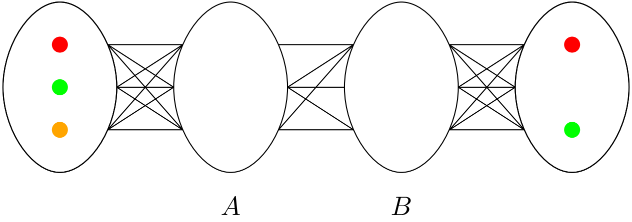

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Within the list-coloring context, clique cutsets are not as convenient as for the classical coloring.
We still manage to deal with them by using **Galvin's theorem.**

Coloring (P_6, bull)-free graphs

Theorem ***Garey, Johnson, Stockmeyer 1974***

Deciding wether a graph is k-colorable is NP-complete for each $k \geq 3$.

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As it is well known, the k-coloring problem is polynomial for perfect graphs.

But Let us take a look at **other graph classes**.

Coloring (P_6, bull)-free graphs

Theorem *Kamiński, Lozin 2007*

For any fixed $k, g \geq 3$, the k-coloring problem is NP-complete in the class of graphs with ***girth*** at least g.

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By combining these two theorems, we have the following :

Corollary

For any fixed $k \geq 3$ and H a forbidden induced subgraph that is **not a collection of paths**, deciding whether a H-free graph is k-colorable is NP-complete.

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k-coloring of P_ℓ-free graphs.

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≤ 4	*P*	*P*	*P*	*P*
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6	*P*	*P*	**?**	*NPC*
7	*P*	*P*	*NPC*	*NPC*
≥ 8	*P*	**?**	*NPC*	*NPC*

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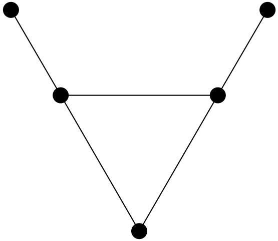
$\vdots$

Theorem *Chudnovsky, Spirkl, Zhong 2018*

There exists a polynomial time algorithm for the 4-coloring problem for P_6-free graphs.

Theorem ***Maffray, Pastor***

There is polynomial time algorithm that determines whether a (P_6, bull)-free graph is 4-colorable, and if it is, produces a 4-coloring.



Coloring (P_6, bull)-free graphs

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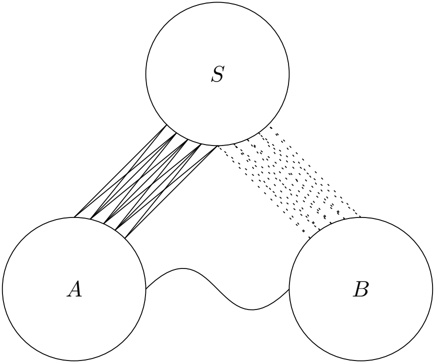

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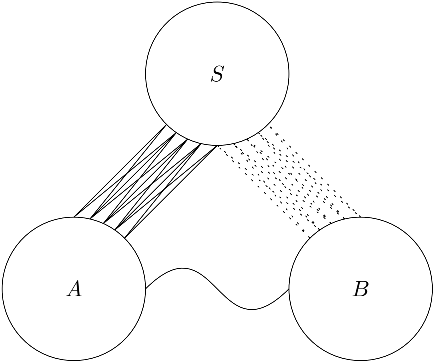

Definition

Quasi-prime graph A graph G is ***quasi-prime*** if every non-trivial homogeneous set of G is a clique.

Coloring (P_6, bull)-free graphs

Lemma

It is sufficient to produce a 4-coloring for any (P_6, bull)-free graph G that satisfies the following properties :

1. G is K_5-free and double-wheel-free.
2. G and $\overline{G}$ are connected.
3. G is quasi-prime.

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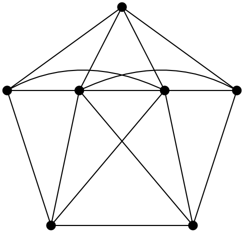

double-wheel

Proof of G and $\overline{G}$ connected.

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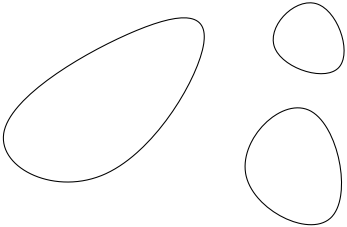

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If G is not connected.

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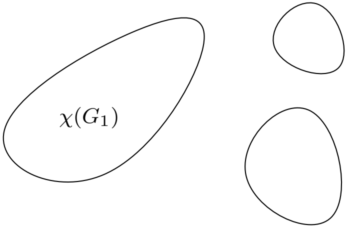

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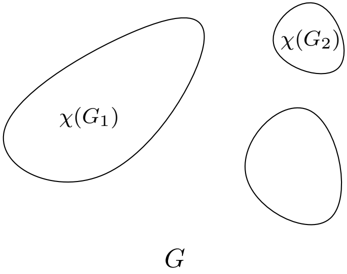

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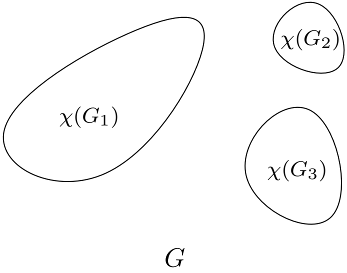

If G is not connected.
Keep the maximum over all $\chi(G_i)$.

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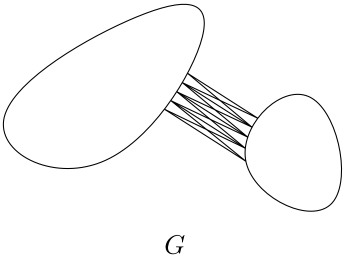

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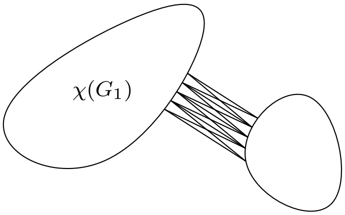

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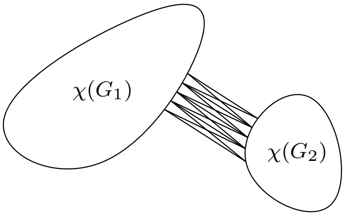

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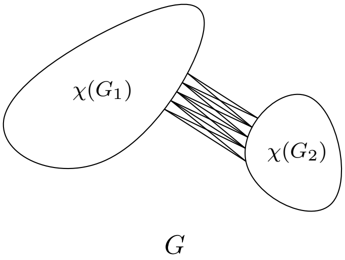

If $\overline{G}$ is not connected.
Test 3-colorability of each co-components with known algorithms.
Refine to test whether they are 1-, 2- or 3-colorable.

Lemma

Let G be a quasi-prime bull-free graph that contains no K_5 and no double-wheel. Then at least one of the following holds :

1. G is gem-free.
2. G contains a ***magnet***.
3. G contains the special graph.

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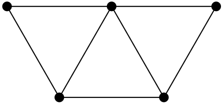
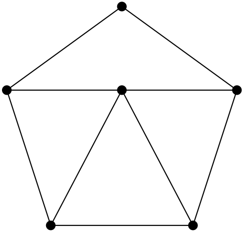


gem



special graph

Coloring (P_6, bull)-free graphs

Lemma

Let G be a (P_6, bull, gem)-free graph, then G is either :

1. Perfect.
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1. Algorithm for bull-free perfect graphs.
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Definition

A subgraph F of G is a ***magnet*** if every vertex of $G \setminus F$ has two neighbours $u, v \in V(F)$ such that $uv \in E(F)$.

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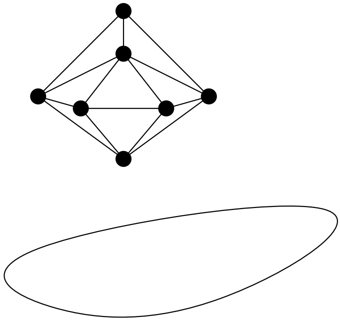

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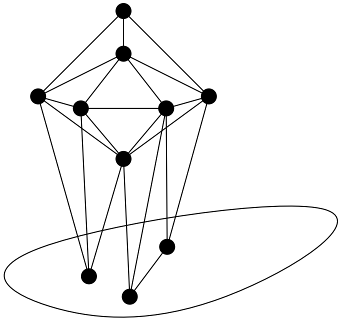

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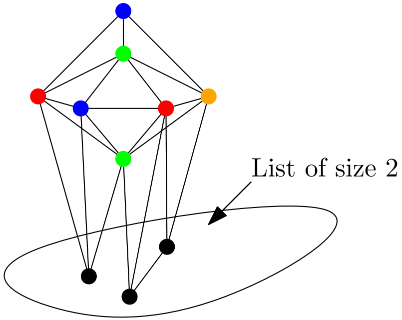

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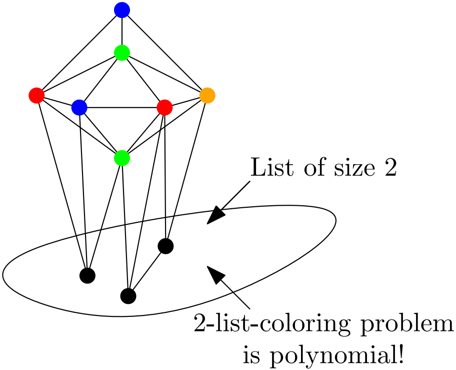

Lemma

Let G be a quasi-prime bull-free graph that contains no K_5 and no double-wheel. Then at least one of the following holds:

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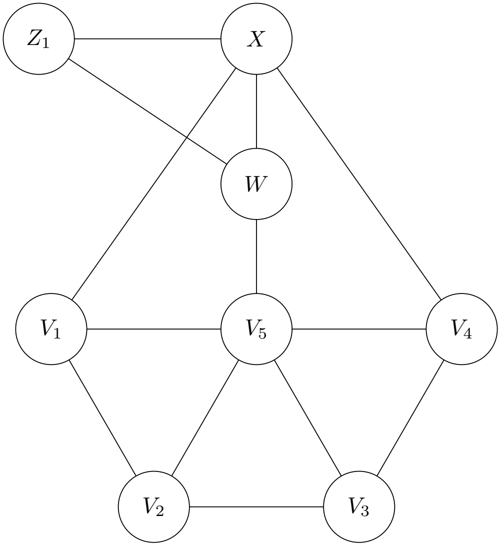
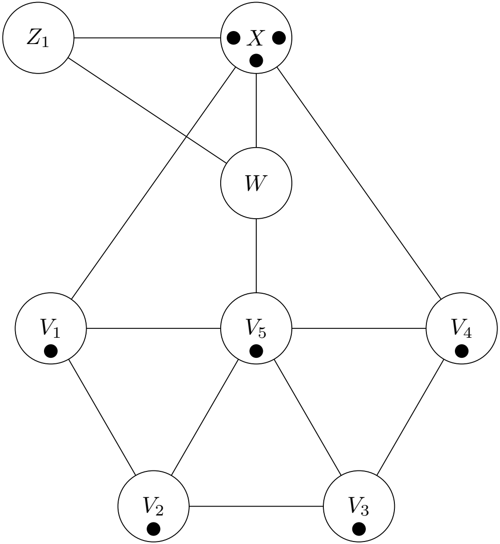
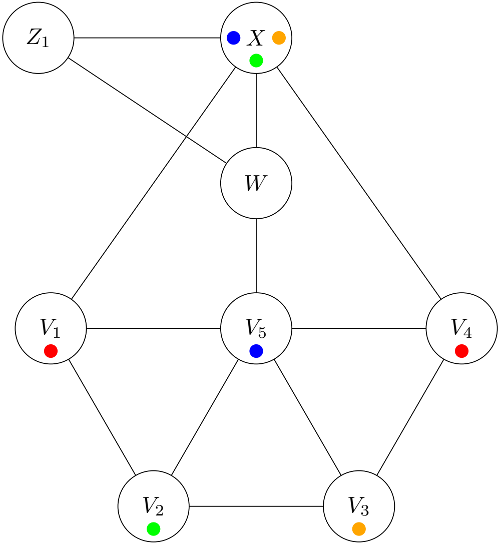
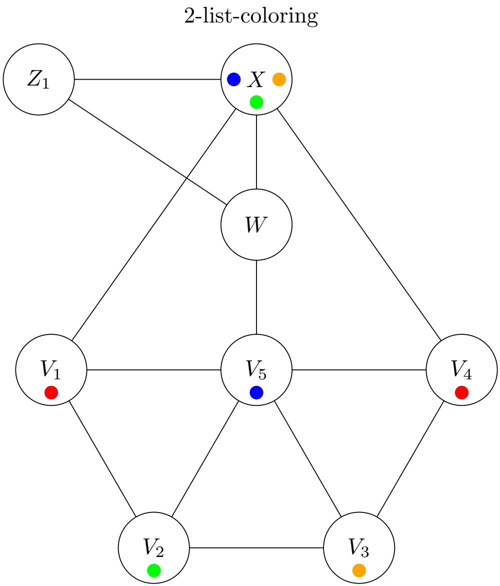

Z_1
X
W
V_1
V_5
V_4
V_2
V_3

Coloring (P_6, bull)-free graphs





Z_1
X
W
V_1
V_5
V_4
V_2
V_3


2-list-coloring
Z_1
X
W
V_1
V_5
V_4
V_2
V_3

Coloring (P_6, bull)-free graphs

Theorem ***Maffray, Pastor***

There is a polynomial time algorithm that determines whether a (P_6, bull)-free graphs is 4-colorable, and if it is, produces a 4-coloring.

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Theorem ***Maffray, Pastor***

For any fixed k, there is a polynomial algorithm that determines if a (P_6, bull, gem)-free graph is k-colorable and if it is, produces a k-coloring.

Conclusion

- At the beginning of my PhD, I realised I knew close to nothing, and he knew **A LOT**.
- During my PhD thesis, I participated in 6 (3 with only the two of us) papers where Frédéric was also a co-author.
- These 3 years of PhD were a walk in the park thanks to him.

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Thank you for your attention.