# Containment graphs and posets of paths in a tree: wheels and partial wheels

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**Abstract**

We consider questions regarding the containment graphs of paths in a tree (CPT graphs), a subclass of comparability graphs, and the containment posets of paths in a tree (CPT orders). In 1984, Corneil and Golumbic observed that a graph $G$ may be CPT, yet not every transitive orientation of $G$ necessarily has a CPT representation, illustrating this on the even wheels $W_{2k}(k \geq 3)$. Motivated by this example, we characterize the partial wheels that are containment graphs of paths in a tree, and give a number of examples and obstructions for this class. Our characterization gives the surprising result that all partial wheels that admit a transitive orientation are CPT graphs. We then characterize the CPT orders whose comparability graph is a partial wheel.

## 1 Introduction

An undirected graph $G$ is a containment graph if each vertex $v_i \in V(G)$ can be assigned a subset $S_i$ of a given set $\mathbf{S}$ such that two vertices $v_i$ and $v_j$ are adjacent if one of their sets strictly contains the other, that is, $(v_i, v_j) \in E(G) \iff S_i \subset S_j$ or $S_j \subset S_i$, where the symbol $\subset$ denotes *proper* inclusion. The containment graphs of arbitrary sets are equivalent to the family of comparability graphs, i.e., those admitting a transitive orientation (TRO), as observed first by Dushnik and Miller [DuMi41]. For this general case, Golumbic and Scheinerman [GoSc89] showed that the subsets may be assumed to be substars of a star.

For the case of containment graphs of intervals on a line, Dushnik and Miller proved that the corresponding subfamily of comparability graphs are those having partial order dimension

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at most 2 [DuMi41], which are known to be equivalent to the family of permutation graphs. Golumbic and Scheinerman [GoSc89] generalized this further, showing that a partial order has dimension at most $2d$ if and only if it is representable as the containment of boxes in $\mathbb{R}^d$ with edges parallel to the axes. The corresponding subfamily of containment graphs are called (rectilinear) *box containment graphs* in $d$-space.

Containment graphs of circular arcs have been studied in [NiMaNa88], where they are shown to be equivalent to the circular permutation graphs of [RoUr82], see also [Sp03]. Circle orders (containment of circular disks in the plane) have been extensively studied [Fi88, Fi89, Sc91, Sc92, ScWi88, SiSiUr88] as well as their 3-dimensional analogue sphere orders [BrWi89, ElFa98, FeFiTr99, Fo93, Sc93]. A survey of these and other known results on geometric containment orders can be found in Fishburn and Trotter [FiTr98].

In this paper, we investigate the containment graphs of paths in a tree (CPT graphs) and some properties of the posets defined by their transitive orientations. In 1984, Corneil and Golumbic [CoGo84]) observed that a graph $G$ may be CPT, i.e., have a containment representation of paths in a tree, yet not every transitive orientation of $G$ necessarily has such a representation, (unlike poset dimension, interval orders, box containment orders and others which are comparability invariants.) For example, looking at the 8-wheel $W_8$ (Figure 1), they showed that the transitive orientation $F$ where the central vertex is a *sink* (interpreted as its path being *contained in* all the paths of the outer vertices) has a representation as the containment order of paths in a tree, but the dual transitive orientation $F^d$ where the central vertex is a *source* (interpreted as its path *containing* all the paths of the outer vertices) has no such representation: namely, if the central vertex corresponds to a path that contains the remaining eight paths, then we would have an interval containment representation for the chordless 8-cycle $C_8$, which is not possible [DuMi41].

Motivated by this example, in Section 3, we study the partial wheels (wheels with missing spokes) that are containment graphs of paths in a tree and provide a characterization for them (Theorem 1). A *partial wheel* consists of a chordless cycle and a central vertex adjacent to *some* but not all of the cycle vertices. Since a containment graph must always be a comparability graph, one might ask first *which partial wheels have a transitive orientation*, and then ask which of those admit a CPT representation. Our characterization gives the surprising result that all partial wheels that admit a TRO are CPT.

In Section 4, we characterize the CPT orders whose comparability graph is a partial wheel (Theorem 2). These results provide us with a characterization of the partial wheels for which every transitive orientation is a CPT order. We conclude with open questions and further directions for research.

# 2 Preliminaries

## 2.1 Definitions and notation

Given an undirected graph $G = (V, E)$, we use the notation $(x, y)$ for an edge, and say that vertices $x$ and $y$ are *adjacent* or *neighbors*. The set of neighbors of a vertex $x$ is denoted by

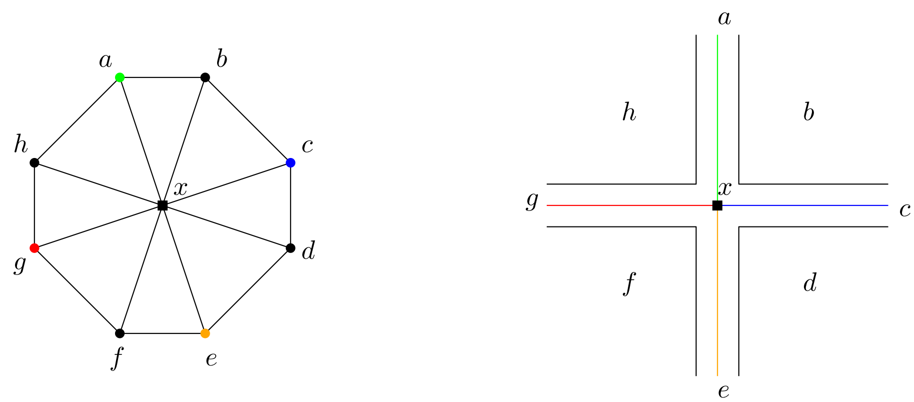


Figure 1: The 8-wheel $W_8$. Its central vertex must be a sink in any CPT representation.

$\mathcal{N}(x)$.

An undirected graph $G = (V, E)$ is a *comparability graph* if it admits a transitive orientation (TRO) of its edges, that is, if $a \to b$ and $b \to c$, then $a \to c$ in the orientation. The transitive orientation is a (strict) partial order on the vertices, and $G$ is its comparability relation.

A fundamental notion in the study of comparability graphs is the $\Gamma$-forcing relation for transitive orientations which can be stated as follows:

> If $(u, v), (v, w) \in E$, but $(u, w) \notin E$, then in any transitive orientation of $G$, orienting $u \to v$ forces the orientation $w \to v$, and orienting $v \to u$ forces the orientation $v \to w$.

Let $\mathcal{S} = \{S_x \mid x \in X\}$ be a family of subsets of a given set $Y$. A partially ordered set $P = (X, \prec)$ (also called a poset) is called a *containment order* with representation $\mathcal{S}$ if $x_i \prec x_j \iff S_i \subset S_j$, where the symbol $\subset$ denotes *proper* inclusion. We call $\mathcal{S}$ a containment representation of the poset.

Let $\mathcal{C}$ denote a class of objects, such as intervals on a line, subtrees or paths of a tree, arcs on a circle, circular disks in the plane, rectilinear boxes in $\mathbb{R}^m$, etc. We call $P = (X, \prec)$ a $\mathcal{C}$-*containment order* if it admits a containment representation $\mathcal{S}$ where all the sets are taken from the class $\mathcal{C}$. Thus, we may speak of interval containment orders, subtree containment orders, circular-arc containment orders, etc. We call $G$ a $\mathcal{C}$-*containment graph* if it admits a transitive orientation which is a $\mathcal{C}$-*containment order*, and speak of interval containment graphs, subtree containment graphs, circular-arc containment graphs, etc.

We will be concerned in this paper with the containment graphs of paths in a tree (CPT graphs) and some properties of the CPT orders defined by their transitive orientations. For example, it is a simple exercise left to the reader that all trees are CPT graphs. Bipartite graphs, however, are not necessarily CPT graphs.

A poset property $\Pi$ is called a *comparability invariant* if, for any given comparability graph $G$, either every transitive orientation of $G$ satisfies property $\Pi$, or no transitive orientation of $G$ satisfies $\Pi$. Many familiar classes of poset properties are comparability invariant, including: poset dimension (see [Tr92]), interval dimension [HaKeMo91], unit interval orders [Go77], box containment orders [GoSc89], bounded tolerance, bitolerance orders, unit tolerance and unit bitolerance orders [BoIsLaTr01]. See also the book [GoTr04].

However, Corneil and Golumbic [CoGo84], observed that the property of being a CPT order is *not* a comparability invariant, as demonstrated by the wheel $W_{2k}$ ($k \geq 3$). In this paper, we will explore this further by characterizing the CPT orders whose comparability graph is a partial wheel (a wheel with missing spokes), and the partial wheels that are CPT graphs.

### 2.2 Interval containment representations for chordless paths

We mention here an important property of interval containment representations for the chordless path $P_n$ that will be used later.

**Remark 1.** *Let $F$ be the (alternating) transitive orientation of an odd chordless path on vertices $[a_1, a_2, \ldots, a_{2j+1}]$ with $a_1$ and $a_{2j+1}$ both sinks. We note that $F$ has an "almost unique" containment representation as a group of intervals on the line:*

> The *odd* numbered intervals (which may overlap or be disjoint) appear on the line with the ordering of their left endpoints being the same as the ordering of their right endpoints, and this will be either their original order on the path or their reverse order. We will call these the "*small*" intervals, and the first and the last will be called the "*extreme*" intervals. (When the set is of size 1, it is the only "extreme" interval.) The *even* numbered interval corresponding to $a_{2i}$ must contain, in sequence, the two small intervals corresponding to its neighbors $a_{2i-1}$ and $a_{2i+1}$ on the path.

**Remark 2** (Compressed intervals-on-line)**.** *Among all the possible representations in Remark 1, there is one canonical endpoint sequence that we will call the* compressed representation of intervals*. It has the property that all intervals have a common point $p$ on the line, by starting all intervals before closing any of them. Namely:*

$$l_1 = l_2 < l_3 = l_4 < l_5 \ \cdots \ l_{2j-1} = l_{2j} < l_{2j+1} < p \text{ and}$$
$$p < r_1 < r_2 = r_3 < r_4 = r_5 \ \cdots \ r_{2j-1} < r_{2j} = r_{2j+1}$$

*where we denote the interval $I_i = [l_i, r_i]$.*

It is easy to see that this will represent the chordless path.

# 3 Partial wheels as containment graphs of paths in a tree

In this section, we raise and answer the question of characterizing the partial wheels that are containment graphs of paths in a tree (CPT). Since the property of being CPT is not a comparability invariant, one might ask first, *which partial wheels have a TRO*, and then ask which of those admit a CPT representation. We show, in fact, that they are equivalent!

**Theorem 1.** *Let $W$ be a partial wheel. The following conditions are equivalent:*

*(1) $W$ has a transitive orientation,*

*(2) $W$ is a containment graphs of paths in a tree,*

*(3) the outer-cycle of $W$ is of even length, and either*

*(a) the central vertex is adjacent to exactly two consecutive outer vertices, or*

*(b) all maximal sets of consecutive neighbors and of consecutive non-neighbors of the central vertex are of odd length.*

*Proof.* Let $\{a_1, a_2, \ldots, a_n\}$ be the outer-cycle of the partial wheel $W$, and denote by $x$ the central vertex, adjacent to *some* but not all of the outer vertices $a_i$.

(1) $\Rightarrow$ (3). If $W$ has a transitive orientation, then in such an orientation, consecutive edges of the outer-cycle alternate direction. Hence, the cycle has even length.

(a) Suppose the central vertex $x$ is adjacent to exactly two consecutive outer vertices, say $a_1$ and $a_2$. We may assume, without loss of generality, that $a_1 \to a_2$. Then, by the standard "$\Gamma$-forcing" relation for comparability graphs, $a_1 \to a_n$, $a_3 \to a_2$ and so $a_1 \to x$, $x \to a_2$. This transitive orientation shows that case (a) is possible.

(b) Suppose that $x$ is *adjacent* to a maximal set of size two, say $a_1, a_2$ with edge orientations as in case (a). If $x$ were to be adjacent to another outer vertex $a_j$ $(4 \leq j \leq n-2)$ then the orientation of the edge $(x, a_j)$ would contradict transitivity with either $a_1 \to x$ or $x \to a_2$. Thus we would have exactly case (a).

Suppose that $x$ is *adjacent* to a maximal consecutive set of neighbors of even length greater than or equal to 4, say $a_1, a_2, \ldots, a_{2i-1}, a_{2i}$ $(i \geq 2)$. Again, we may assume, without loss of generality, that $a_1 \to a_2$. Because of the parity on the outer-cycle, this forces $a_1 \to x$, $x \to a_{2i}$ which contradicts transitivity. Thus, we may now assume that aside from case (a), all consecutive sets of neighbors of $x$ are of odd length.

Suppose that $x$ is *non-adjacent* to an even length consecutive gap of non-neighbors, $a_{j+1}, \ldots, a_{j+2i}$ $(i \geq 1)$ where $x$ is adjacent to $a_j$ and to $a_{j+2i+1}$. Again, by a parity argument on the alternating orientation of the outer vertices, since the length $2i$ of the gap is even length, there will be a violation of transitivity. Thus, all consecutive sets of non-neighbors of $x$ are of odd length.

(3) $\Rightarrow$ (2). We construct an appropriate CPT representation in each case. We begin with the representation of the outer-cycle and its paths.

Let $T$ be a star with center $c$ and edges $\{(c, b_1), (c, b_2), \ldots, (c, b_k)\}$ where $n = 2k$. Let the path $P_1$ consist of the two star-edges $\{(c, b_k), (c, b_1)\}$ representing outer-cycle vertex $a_1$, and let the path $Q_1$ consist of the single star-edge $\{(c, b_1)\}$ representing outer-cycle vertex $a_2$. Similarly, for $i = 2, \ldots, k$, let $P_i = \{(c, b_{i-1}), (c, b_i)\}$ representing $a_{2i-1}$, and let $Q_i = \{(c, b_i)\}$ representing $a_{2i}$. Clearly, $\mathcal{T} = \{P_i \,|\, i = 1, \ldots, k\} \cup \{Q_i \,|\, i = 1, \ldots, k\}$ is a CPT representation of the outer-cycle.

**Case A:** Assume that $x$ is adjacent only to $a_1$ and $a_2$. We extend the representation $\mathcal{T}$ to obtain $\mathcal{T}'$ by $(i)$ adding a new pendant vertex $d$ adjacent to $b_1$ in $T$, $(ii)$ adding the edge $(b_1, d)$ to $P_1$ and $P_2$, and $(iii)$ adding the path $R_x$ consisting of the two edges $\{(c, b_1), (b_1, d)\}$ representing $x$, center of the wheel. This is a CPT representation for Case A.

**Case B:** The construction for this case is a bit more complicated, but has a similar flavor. Start with the same representation $\mathcal{T}$ of the outer-cycle. This time, let $R_x$ be $c$, the center of the star. We now have a representation of the full wheel. Let's modify the representation to "erase" the unwanted edges and get a representation $\mathcal{T}''$ of our partial wheel.

We may assume that the first sequence of consecutive neighbors of $x$ in $W$ is $a_1, \dots, a_{2j-1}$ for some $j \geq 1$. Consider the first gap of *non-neighbors* $a_{2j}, \dots, a_{2j+2i}$ $(i \geq 0)$.

$(i)$ If $i = 0$, shorten $Q_j$ by pulling it back away from the center $c$. Thus, $a_{2j}$ is no longer adjacent to $x$, but it maintains its outer-cycle neighbors.

$(ii)$ If $i \geq 1$, we make the following transformation:

Consider the rooted subtree consisting of tree-edges $\{(c, b_j), \dots, (c, b_{j+i})\}$.

- Pull back (away from the center $c$) this subtree, and together with the "fan" of paths $Q_j, P_{j+1}, \dots, Q_{j+i}$.
- Join the root $c'$ of the subtree to $c$ by a new tree-edge $(c, c')$.
- Reconnect the pieces of $P_{j-1}$ and $P_{j+i+1}$ by adding $(c, c')$ to each of them.

Continue doing this for each gap of *non-neighbors* of $x$. See Figure 2 for an example. It is easy to verify that this will give a CPT representation of $W$.

(2) $\Rightarrow$ (1). Trivial.

□

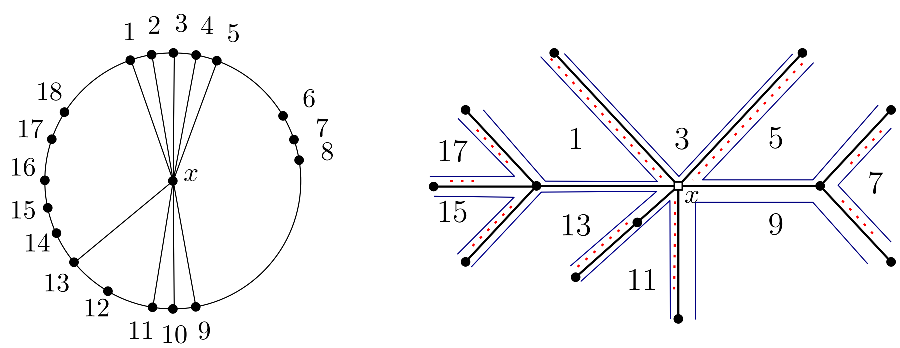


Figure 2: The transformation in (3) $\Rightarrow$ (2), Case B$(ii)$. The $P_i$ paths are indicated in solid blue, and the $Q_i$ paths are indicated in dotted red.

**Remark 3.** *It is easy to see that, in Case A, the central vertex $x$ must have in-degree 1 and out-degree 1. The only other case where $x$ can have both an incoming and an outgoing edge is when $x$ is adjacent to exactly 3 consecutive outer vertices $a_1, a_2, a_3$: $x$ is a true twin of $a_2$, and the edge $(x, a_2)$ can be oriented in either direction.*

*In Case B, without loss of generality, $x$ may be assumed to be a sink (interpreted as its path being contained in all the paths of its neighbors). However, there are many possible partial wheels that have a CPT representation where $x$ is a source, as we will show in the next section.*

# 4 When can the central vertex of a partial wheel be a source?

We will now characterize the CPT representations of partial wheels in the case where the central vertex $x$ is a source, that is, where the path $P_x$ contains all the paths $\mathcal{P}_{\mathcal{N}(x)} = \{P_{a_i} \mid a_i \in \mathcal{N}(x)\}$ of its neighbors. This will allow us to characterize all CPT orders of partial wheels.

Corneil and Golumbic [CoGo84] observed that since the subgraph $W_{\mathcal{N}(x)}$ induced by the neighbors $\mathcal{N}(x)$ of a source $x$ is necessarily an interval containment graph, many impossible configurations may arise. For example, in the case of the full wheel $W_{2k}$, $k \geq 3$, the neighborhood of $x$ induces the chordless cycle $C_{2k}$ which is not an interval containment graph. Therefore, *in any CPT representation of a graph $G$, the central vertex of every full wheel of size at least 6, must be a sink with respect to its outer neighbors.* For partial wheels, the situation is more interesting.

**Theorem 2.** *For wheels and partial wheels, the following characterizes their containment orders of paths in a tree.*

*(1) For the full wheel $W_{2k} (k \geq 3)$, the only transitive orientation which is CPT is that with the central vertex as a sink.*

*(2) For an even length partial wheel with the central vertex adjacent to exactly 2 consecutive vertices, there are two transitive orientations and both are CPT.*

*(3) For an even length partial wheel with the central vertex adjacent to exactly 3 consecutive vertices, there are four transitive orientations and all are CPT.*

*For any other partial wheel $W$ of even length at least 6 satisfying condition (3)(b) of Theorem 1, we have the following:*

*(4) If the gaps of $W$ are all of length 1, then the only transitive orientation which is CPT is that with the central vertex as a sink.*

*(5) Otherwise, there are two transitive orientations: with the central vertex as either a sink or as a source, and both are CPT.*

Before proving this result, we present several examples of its consequences.

**Example 1.** *Referring to Figure 3:*

**The bipartite wheel $BW_{2k}(k \geq 3)$ is CPT:** *The bipartite wheel consists of the outer-cycle $\{a_1, a_2, \ldots, a_{2k}\}(k \geq 3)$, and all the odd numbered edges $(x, a_{2i-1})$ for $(1 \leq i \leq k)$ from the center to the outer vertices. It has a CPT representation whose center must be a sink by (4).*

**The 4-crown is not CPT:** *Each vertex is the center of a bipartite 6-wheel $BW_6$. Thus, each vertex must be a sink, which is impossible.*

**The 3×4 grid is not CPT:** *The two inside vertices are both centers of an induced $BW_8$, thus forcing both to be sinks, which is impossible.*

**A dumbbell graph is not CPT:** *Trying to join two full wheels (or two bipartite wheels) by an additional single edge connecting their centers, would be a "dumbbell" idea for someone trying to build a CPT graph automobile axle. That edge would have two sinks.*

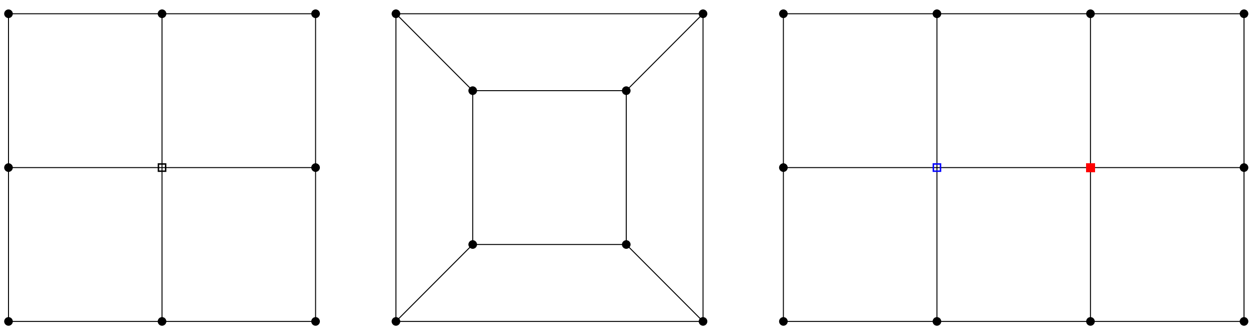

Figure 3: (a) The bipartite 8-wheel $BW_8$. (b) The 4-crown ($X_{79} rostock$). (c) The 3×4 grid.

*Proof of Theorem 2.* For statement (1), the result is due to Corneil and Golumbic [CoGo84], as mentioned earlier.

Let $W$ be a partial wheel with center vertex $x$ and outer-cycle $\{a_1, a_2, \ldots, a_{2k}\}(k \geq 3)$ in a clockwise direction. We have already seen in Remark 3, that statements (2) and (3) hold, and that in all other cases, $W$ has a CPT representation with the central vertex as a sink.

Consider the transitive orientation $F$ of $W$ with $x$ as a source. We will prove statement (4) by giving a contradiction, and give a constructive proof of statement (5), after presenting some terminology and observations.

By renumbering, we may assume that $a_{2k}$ is non-adjacent to $x$ and $a_1$ is adjacent to $x$. Thus, by the properties of transitive orientations and by condition (3b) of Theorem 1, we know that

(i) the odd numbered vertices are sinks, forming an independent set, and

(ii) every consecutive set of neighbors of $x$ begins with a sink and ends with a sink.

Moreover, by the observation of Cornell and Golumbic [CoGo84],

(iii) in any CPT representation of $F$, the paths of the neighbors of $x$ must be *intervals* on the path $P_x$, which we will call "the line".

Summarizing, as in Remark 1, this entire group of intervals will appear on the line either in the same order as their clockwise ordering in $W$, or the opposite flipped ordering. Then the extreme intervals between groups will have to be joined by the "gap" intervals according to their adjacencies on the cycle, as we will see shortly.

**Note**: *In principle, these groups of neighbors could be placed randomly on the line, but their placement will be restricted by the sizes and distribution of the paths of the gap vertices that need to connect them.*

For example, the following is clear:

**Claim 1**. *Let two groups of consecutive sets of neighbors appear one after the other on the outercycle. If the gap between them consists of only one non-neighbor $a_{2j}$, then its path $P_{a_{2j}}$ must contain the small extreme intervals $P_{a_{2j-1}}$ and $P_{a_{2j+1}}$ and then must branch off the line since it is a non-neighbor of $x$.*

We are now ready to prove statement (4) of the theorem: Suppose all the gaps of non-neighbors of $W$ are of length 1. If $W$ has only one such gap $a_{2k}$, then it is impossible for $P_{a_{2k}}$ to contain both extreme intervals without containing all the small intervals. If there are several such gaps, then by our Claim 1, the groups of consecutive sets of neighbors would have to be laid out on the line in exactly the same order they appear on the outercycle. But this will be impossible since the last gap path will not be able to contain the remaining pair of extreme intervals without containing all the small intervals. This proves statement (3). Moreover, in particular, it shows that *for $BW_{2k} (k \geq 3)$ the center must be a sink.*

For statement (5) of the theorem, we will now provide a construction for all even partial wheels with at least one gap that is longer than 1:

Let us assume that the last gap has length 3 or larger. Our construction will gather together the groups of consecutive neighbors separated by gaps of size exactly 1, which we will call *stretches*. We will lay out the stretches on the line in an alternating manner.

Formally, we define a *stretch* to be a maximal length sequence of outer vertices whose non-neighbor gaps are all of length 1. By (ii) above, we also have that

(iv) every stretch begins with a sink and ends with a sink, and they will be the extreme vertices of the stretch, and

(v) the number of stretches equals the number of gaps of length 3 or more.

**Step 1** (*representing a stretch*): Since a stretch $\{a_{2i+1}, \ldots, a_{2j+1}\}$ induces a chordless path, we will first take a compressed interval representation of this chordless path as in Remark 2, and then for each gap vertex, extend its interval downward out of the line, since it is a non-neighbor of $x$, as illustrated in Figure 4. It is easy to see that this will represent the stretch.

**Step 2** (*laying out the stretches*): Let the stretches be numbered clockwise $\{S_1, \ldots, S_m\}$

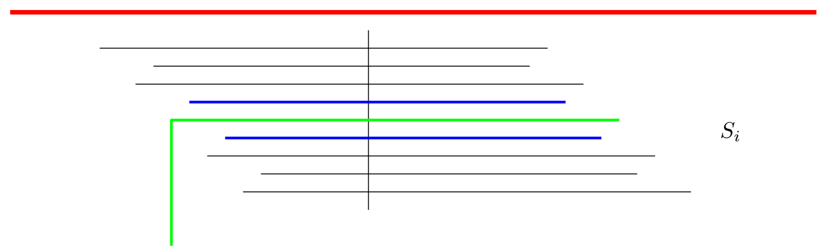


Figure 4: A stretch with a gap of size 1. The gap is represented by the bent path (in green). The bold intervals (in blue) represent the two neighbors of the gap in the stretch. The top interval (in red) is the path representing the central vertex of the wheel.

and let $p_t$ be the "common point" of the intervals of stretch $S_t$.

We lay out the stretches in the following order, where $S_t^{-1}$ means in reversed (flipped) order, as illustrated in Figure 5:

If $m$ is odd, $\quad S_1,\ S_m^{-1},\ S_2,\ S_{m-1}^{-1},\ S_3,\ \cdots\ ,S_{(m+1)/2+1},\ S_{(m+1)/2}.$
If $m$ is even, $\quad S_1,\ S_m^{-1},\ S_2,\ S_{m-1}^{-1},\ S_3,\ \cdots\ ,S_{m/2},\ S_{m/2+1}^{-1}.$

**Step 3** (*connecting the stretches*): We connect ascending stretches, like $S_1$ and $S_2$, which have the stretch $S_m^{-1}$ separating them on the line, with a sequence of "gap" paths $P_{2i}, \ldots, P_{2j}$ as follows:

(I) Add a "fan" of $t$ additional pendant edges $e_1, \ldots, e_t$ at the common point $p_m$ of $S_m^{-1}$, where $2t + 1 = 2j - 2i + 1$ is the length of the gap between $S_1$ and $S_2$.

(II) $P_{2i}$ starts with the left endpoint of the right extreme interval of $S_1$, continues to the common point $p_m$ and branches off-the-line onto the new leg of the tree $e_1$.

(III) $P_{2j}$ starts with the new leg $e_t$ and continues on the line to the right endpoint of the left extreme interval of $S_2$.

(IV) Fill in the fan with wedges and small intervals to realize the chordless path of the gap sequence, as in the "star and pizza" construction in the proof of Theorem 1. (An illustration of this process is presented in Figure 6).

We connect descending stretches in a similar matter. The transition from ascending to descending, at the end of the line, is also handled in a similar manner.

If there are at least two gaps of length greater than 3, it is a simple matter to verify that this construction gives a CPT representation for the partial wheel $W$ with $x$ as a source. If there is only one gap of length greater than 3, and therefore only one stretch, we simply modify the representation of that stretch by making the two extreme intervals meet in the common point (instead of overlap), that is, $l_{2j+1} = p = r_{2i+1}$ in Remark 2, and insert the same kind of "fan" to connect the two extreme intervals.

This concludes the proof of the theorem. $\square$

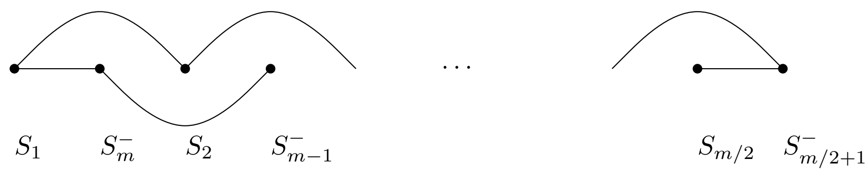


Figure 5: The layout of an even number of stretches when the central vertex of the wheel is a source. Each vertex represents a stretch or its reversal.

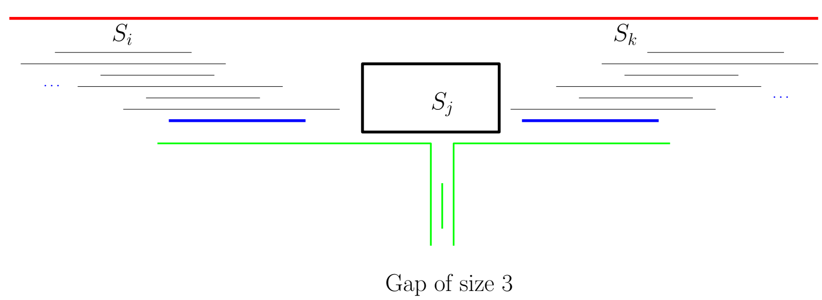


Figure 6: Realization of the partial wheel when the central vertex of the wheel is a source and when the gap is of size 3. The bold intervals (in blue) represent the extremities of the two stretches that are connected to the gap. The paths below them (in green) correspond to the vertices of the gap. The top interval (in red) is the path representing the central vertex of the wheel.

# 5 Open Problems

In this paper, our focus has been to characterize the partial wheels having containment representations of paths on a tree. These were used to obtain a number of minimal forbidden subgraphs and orders in the general case, including infinite families of such minimal obstructions. Many open questions remain.

## 5.1 What are the CPT graphs and orders?

The problems of characterizing the CPT graphs and the CPT orders remain as open questions. The same is true for bipartite CPT graphs. We may also ask for which CPT graphs will all transitive orientations admit CPT containment representations, and like in the case of full wheels, for which other comparability graphs will only one TRO be CPT and not its reversal.

A CPT order $P$ is called *dually-CPT* if both $P$ and its dual $P^d$ are CPT orders. For example, in our Theorem 2 statements (2) and (4) together characterize the dually-CPT orders of partial wheels. Alcon, et al. [AlGuGu18] have asked whether the poset dimension of dually-CPT orders is bounded above by a constant. If so, this would be a generalization of the known result for interval containment orders which have poset dimension 2.

Alcón, et al. [AlGuGu18] also gave a characterization of CPT split orders by a family of forbidden subposets. Similarly, questions of characterization and complexity can be asked about other subfamilies of CPT graphs and orders.

Spinrad [Sp03] asked whether the poset dimension of CPT orders have a constant upper bound. Alcón, et al. [AlGuGu18] showed that this is not the case: the poset dimension of CPT orders is unbounded, however, it is at most the number of leaves of the host tree used in the containment model. Majumder, Mathew and Rajendraprasad [MaMaRa18] give an asymptotically tight bound on the dimension of a CPT poset, in terms of the maximum degree and radius of the host tree, which is tight up to a multiplicative factor of $(2+\epsilon)$, where $0 < \epsilon < 1$. It was also pointed out by an anonymous referee that since the order induced by levels 1 and 2 of a Boolean lattice is a CPT order, the dimension of the class is unbounded.

## 5.2 A comment on: containment versus proper containment

We have defined containment graphs and containment orders using “proper subset”, “strict partial order” and “proper containment” following [Go84, GoSc89, FiTr98] and others. This allows duplicating vertices as false twins (non-adjacent vertices with equal neighborhoods) by simply assigning the same set to each twin, but it does not always permit duplicating vertices as true twins (adjacent vertices with equal neighborhoods).

This dichotomy is best illustrated when “cloning” the central vertex $x$ of the wheel $W_{2k}$, that is, suppose you add a new vertex $x'$, adjacent to $x$, and to all the outer vertices $a_1, \ldots, a_{2k}$. By our Theorem 2, both $x$ and $x'$ must be sinks within their respective wheels in any CPT representation. This poses no problem for the transitive orientation since the new edge $(x, x')$ could be oriented in either direction. Moreover, when $2k$ itself is divisible by 4, it is possible to have a CPT representation assigning different intervals $x$ and $x'$, one containing the other. However, if $2k$ is not divisible by 4, then one can show that the only way to obtain a CPT representation of $W_{2k}$ is to assign a single node (a point-path) of the tree to the center of the wheel. Thus, it would be impossible to do this for both $x$ and $x'$ if they are adjacent.

# References


[AlGuGu18] L. Alcón, N. Gudiño and M. Gutierrez, Recent results on containment graphs of paths in a tree, *Discrete Applied Math.* 245 (2018), 139–147.

[BoIsLaTr01] K. Bogart, G. Isaak, J. Laison and A. Trenk, Comparability invariance results for tolerance orders, *Order* 18 (2001), 281–294.

[BrWi89] G. R. Brightwell and P. Winkler, Sphere orders, *Order* 6 (1989), 235–240.

[CoGo84] D. Corneil and M. C. Golumbic, unpublished, but cited in [Go84, GoSc89].

[DuMi41] B. Dushnik and E. W. Miller, Partially ordered sets, *Amer. J. Math.* 63 (1941), 600–610.

[ElFa98] M. H. El-Zahar and L. A. Fateen, On sphere orders, *Discrete Math.* 185 (1998), 249–253.

[FeFiTr99] S. Felsner, P. C. Fishburn and W. T. Trotter, Finite three-dimensional partial orders which are not sphere orders, *Discrete Math.* 201 (1999), 101–132.

[Fi88] P. C. Fishburn, Interval orders and circle orders, *Order* 5 (1988), 225–234.

[Fi89] P. C. Fishburn, Circle orders and angle orders, *Order* 6 (1989), 39–47.

[FiTr98] P. C. Fishburn and W. T. Trotter, Geometric containment orders: a survey, *Order* 15 (1998), 167–182.

[Fo93] D. G. Fon-Der-Flaass, A note on sphere containment orders, *Order* 10 (1993), 143–145.

[Go77] M. C. Golumbic, Comparability graphs and a new matroid, *J. Comb. Theo. B* 22 (1977), 68–90.

[Go84] M. C. Golumbic, Containment graphs and intersection graphs, NATO Advanced Institute on Ordered Sets, Banff, Canada, May 1984 (abstract only); IBM Israel Technical Report 88.135, July 1984.

[GoSc89] M. C. Golumbic and E. R. Scheinerman, Containment graphs, posets and related classes of graphs, *Ann. N.Y. Acad. Sci.* 555 (1989), 192–204.

[GoTr04] M. C. Golumbic and A. N. Trenk, *Tolerance Graphs*, Cambridge University Press, 2004.

[HaKeMo91] M. Habib, D. Kelly and R. Möhring, Interval dimension is a comparability invariant, *Discrete Applied Math.* 88 (1991), 211–229.

[MaMaRa18] A. Majumder, R. Mathew and D. Rajendraprasad, Dimension of CPT posets, (https://arxiv.org/abs/1802.09326), submitted for publication, 2018.

[NiMaNa88] M. V. Nirkhe, S. Masuda and K. Nakajima, Circular-arc containment graphs, University of Maryland, Technical Report SRC TR 88-53 (1988).

[RoUr82] D. Rotem and J. Urrutia, Circular permutation graphs, *Networks* 12 (1982), 429–437.

[Sc91] E. R. Scheinerman, A note on planar graphs and circle orders, *SIAM J. Discrete Math.* 4 (1991), 448–451.

[Sc92] E. R. Scheinerman, The many faces of circle orders, *Order* 9 (1992), 343–348.

[Sc93] E. R. Scheinerman, A note on graphs and sphere orders, *J. Graph Theory* 17 (1993), 283–289.

[ScWi88] E. R. Scheinerman and J. C. Wierman, On circle containment orders, *Order* 4 (1988), 315–318.

[SiSiUr88] J. B. Sidney, S. J. Sidney and J. Urrutia, Circle orders, N-gon orders and the crossing number, *Order* 5 (1988), 1–10.

[Sp03] J. P. Spinrad, *Efficient Graph Representations*, Fields Institute Monographs 19, American Mathematical Society, Providence, 2003.

[Tr92] W. T. Trotter, *Combinatorics and Partially Ordered Sets,* Johns Hopkins University Press, Baltimore, 1992.