A survey about Solitaire Clobber

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Abstract

Solitaire Clobber is a one-player variant of the 2-player board game Clobber introduced by Albert et al. in 2002. According to simple rules, the objective of Solitaire Clobber is to capture the maximum number of stones from a given graph. Two versions of Solitaire Clobber were recently investigated : a partisan and an impartial one. In this survey, we give an overview of the major results about Solitaire Clobber, more especially about the impartial version. In particular, the game is considered on grids, trees, hypercubes...Two new results are provided: when playing on a tree, we show that the minimum number of remaining stones can be computed in polynomial time. We also assert that any game position on a "large" grid can be reduced to 1 or 2 stones. Note that in each part of this survey, we propose several open problems related to Solitaire Clobber.

1 Introduction

1.1 2-player versions

In 2001 Albert et al. investigated a new 2-player partisan game called Clobber. They developed the first results in [1]. In terms of game values, it turns out that Clobber is difficult even when played on basic positions. This complexity explains the author's motivation for studying Clobber. The description of the game follows below.

2-PLAYER CLOBBER
Black and white stones are placed on the vertices of an undirected graph, at most one per vertex. The first player moves only black stones and the second player the white ones. A player moves by picking up one of his stones and "clobbering" an adjacent stone of the opposite color (vertically or horizontally). The clobbered stone is deleted and replaced by the one that was moved. The last player to move wins.

Clobber is usually played on a grid where the initial position is the one of a checkerboard, as depicted by Figure 1.

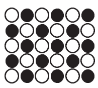


Figure 1: Initial Clobber position on a 5×6 grid

In the last few years, several events were organized around Clobber like the first international Clobber tournament. It was held at the 2002 Dagstuhl seminar on algorithmic and combinatorial game theory (see the report in [12]). Since 2005 the game is one of the events of the Computer Olympiad. A short article for the general public was also written in *Science News* (see [17]).

In [1] it was proved that determining the winner of a Clobber game is a NP-hard question. Moreover, even when the game is played on one or two rows, there are no proved results. Nevertheless the two following conjectures seem to hold :

Conjecture 1 (Albert, Grossman, Nowakowski and Wolfe). *The game position* $(●○)^n$ *is winning for the first player if and only if* $n \neq 3$*.*

Conjecture 2 (Albert, Grossman, Nowakowski and Wolfe). *For* n *odd, the position* $\left(\genfrac{}{}{0pt}{}{●}{○}\right)^n$ *is a first player win.*

When n is even, it is easy to see that the position $\left(\genfrac{}{}{0pt}{}{●}{○}\right)^n$ is a second player win by invoking symmetry arguments.

Problem 1. 2-player Impartial Clobber
Since Clobber appears to be a hard game even on regular positions , what about the impartial version ? What can we say about 2-player Impartial Clobber, where the rules are the same, except that each player can indifferently move a black or a white stone. For example start by studying this game on $1 \times n$ *and* $2 \times n$ *checkerboard positions. Can we provide the same kind of conjecture as for the partisan version ?*

1.2 Solitaire versions

In 2004 Demaine et al. [5] investigated the *Solitaire Clobber* game. It can be seen as the solitaire version of the 2-player partisan Clobber. The way to move the stones is the same as for Clobber, but the objective is adapted for a unique player. Indeed, the goal is to minimize the number of remaining stones on the graph. Here is a full description of this solitaire variant:

Partisan Solitaire Clobber
Black and white stones are placed on the vertices of an undirected graph, at most one per vertex. The player moves by picking up alternately a black and a white stone and "clobbering" an adjacent stone of the opposite color. The clobbered stone is replaced by the one that was moved. The objective consists in minimising the number of remaining stones on the graph.

In [5] Demaine et al. investigated this game on rows and grids with initial "checkerboard position".

Proposition 3 (E.Demaine, M.Demaine and Fleischer). *In Partisan Solitaire Clobber, the reducibility value of a checkerboard row of size* n *is* $\lceil n/4 \rceil$ *if* $n \not\equiv 3 \bmod [4]$*, and* $\lceil n/4 \rceil + 1$ *otherwise.*

The results about checkerboard grids will be developed in Section 2 of this survey. In particular, Demaine et al. showed that such grids are 2-reducible.

A second version of Solitaire Clobber was studied in [7], which corresponds to an impartial 1-player variant. We call it Impartial Solitaire Clobber. The only difference with the above solitaire game is that the player is not forced to alternate black and white captures. Without this constraint, it turns out that Solitaire Clobber becomes easier to work on. Clearly, 1-reducible positions do not need to have the same number of black and white stones (e.g. ●○○○○○, which is $1 - reducible$ for Impartial Solitaire Clobber and not for the Partisan version).

Notice that there exists a natural correlation between the two versions of Solitaire Clobber. Indeed, playing the impartial game on a graph G is equivalent to playing the partisan version on two complementary copies of G. This property is illustrated by Figures 2 and 3, where G is a grid graph (i.e., an induced subgraph of the grid). One play optimally the Impartial Solitaire Clobber on G if and only if one play optimally the partisan game on Figure 3.

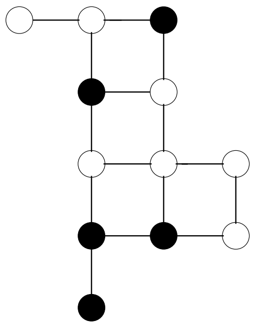


Figure 2: On a grid graph G

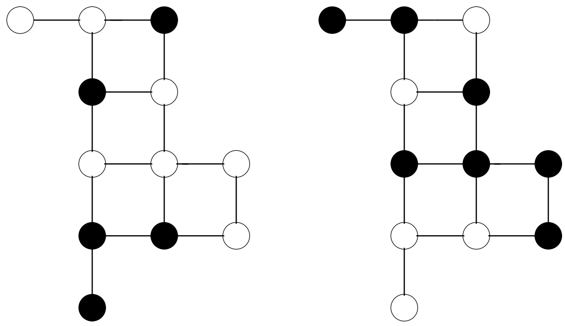


Figure 3: On two complementary copies of G

1.3 Definitions and notations for Solitaire Clobber

We here introduce several definitions and notations that are available for both versions of Solitaire Clobber.

A *game position* on a graph $G = (V, E)$ is a mapping $\Phi : V \rightarrow \{\bullet, \circ\}$.

Given a game position Φ of Solitaire Clobber on a graph G, we say that the **reducibility value** of (G, Φ) is the minimum number of stones that can be left on the graph from initial position Φ. This value is denoted by $rv(G, \Phi)$. We also say that a position on a graph is **k-reducible** (for a positive integer k) if there exists a succession of moves that leaves at most k stones on the graph. In particular, as for the "classical" Solitaire game, we are interested in Solitaire Clobber positions that are 1-reducible.

For short if there is no confusion, we may say that G or Φ is k-reducible instead of (G, Φ) is k-reducible. Similarly, we may confound a vertex with the stone it supports.

If all the vertices of the graph G have the same color, we say that G is **monochromatic**. If there exists a vertex v such that $G \setminus v$ is monochromatic and G is not, then G is said to be **quasi-monochromatic**.

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *Cartesian product* $G_1 \square G_2$ is the graph $G = (V, E)$ where $V = V_1 \times V_2$ and $(u_1u_2, v_1v_2) \in E$ if and only if $u_1 = v_1$ and $(u_2, v_2) \in E_2$, or $u_2 = v_2$ and $(u_1, v_1) \in E_1$. One generally depicts such a graph with $|V_2|$ vertical copies of G_1, and $|V_1|$ horizontal copies of G_2, as shown on Fig. 4.

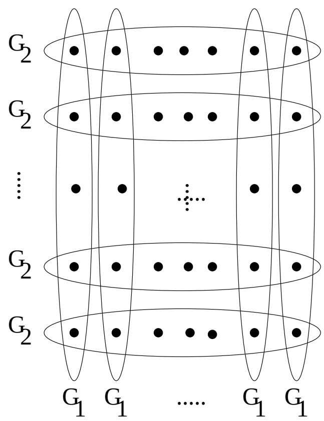


Figure 4: The Cartesian product of two graphs G_1 and G_2

A *grid graph* is an induced subgraph of the $k \times n$ grid $P_k \square P_n$, where P_k and P_n are two paths of respective lengths k and n.

1.4 Complexity results

Whatever the type of the game (2-player or Solitaire) and the way of moving (Impartial or Partisan), it turns out that Clobber is a hard game. The proofs of the NP-hardness often rely on the Hamiltonicity problem on graphs. The correlation between these two problems appears when playing Clobber on a quasi-monochrome position.

The first complexity result was proved by Albert et al. in [1]. Their proof uses a reduction to the HAMILTONIAN PATH problem.

Theorem 4 (Albert, Grossman, Nowakowski, Wolfe). *Determining who wins from a 2-player Clobber position is NP-hard.*

The same kind of result was provided by Demaine et al. in [5] for Solitaire Clobber.

Theorem 5 (E.Demaine, M.Demaine, Fleischer). *Deciding whether or not a position of Solitaire Clobber is* 1*-reducible is NP-complete.*

Their proof can be adapted for both impartial and partisan versions. As for the 2-player game, it requires a reduction to the Hamiltonian path problem. On Figure 5 below it is easy to see that the graph on the left admits an Hamiltonian path between vertices v_1 and v_2 if and only if the position of the right is 1-reducible for Impartial Solitaire Clobber. A small gadget (see [5]) leads to the same kind of reduction for the partisan version.

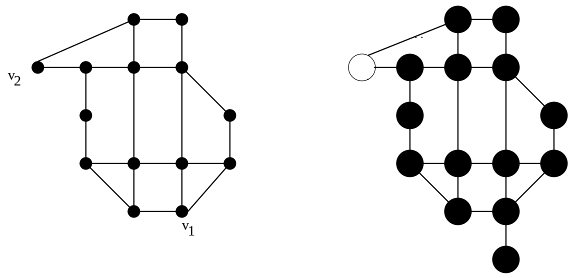


Figure 5: Reduction to Hamiltonian path

By using a result of Itai et al. (see [15]), Demaine et al. showed that the NP-completeness of Solitaire Clobber remains true on grid graphs. In Section 2, we give some elements to solve the complexity on grids. Indeed, we show that for Impartial Solitaire Clobber, non monochromatic positions on grids are 2-reducible. Nevertheless the complexity remains open.

1.5 Organization of the paper

A major part of this survey will be devoted to the investigation of Impartial Solitaire Clobber. Section 2 deals with the game on grids and compares the different results about both versions of Solitaire Clobber. In particular, it is showed that grids are always 2-reducible for Impartial Solitaire Clobber, whatever the (non-monochromatic) initial position is. In this section, some of the stuff introduced by Demaine et al. will be reinvestigated for the study of the impartial game. In Section 3 we present algorithms to compute the reducibility value of game positions on paths, cycles and trees. On such families of graph, this computation can be done in polynomial time. In Section 4 we give constant upper bounds on the reducibility value when playing on Hamming graphs and multipartite complete graphs. Section 5 is devoted to more general results, such as positions that minimize or maximize the reducibility value on an arbitrary graph. Moreover in each section we propose several other combinatorial optimization problems related to Solitaire Clobber.

2 Solitaire Clobber on grids

2.1 Special positions on grids

Checkerboard Positions.

Initially, Clobber was played on a complete grid ($k \times n$ rectangle) with checkerboard position Φ_c, where Φ_c is defined by $\Phi_c(i,j) = \bullet$ if $(i+j)$ is even, and $\Phi_c(i,j) = \circ$ otherwise. In [5], Demaine et al. solved the game on such a position :

Theorem 6 (E.Demaine, M.Demaine and Fleischer). *The $k \times n$ checkerboard is 2-reducible. Moreover, the $k \times n$ checkerboard is 1-reducible if and only if $k.n \not\equiv 0 \bmod [3]$.*

The proof of 2-reducibility is done by induction through local rules. In order to prove that a position is not 1-reducible, Demaine et al. define an invariant δ available for both versions of Solitaire Clobber on bipartite graphs. The description of δ follows.

Definition 7. *Let Φ be a game position on a bipartite graph $G = (V, E)$. Denote by S_0 and S_1 two disjoined independent sets of G such that $S_0 \cup S_1 = V$. We allocate the color white the to set S_0, and the color black to S_1. A stone of G is said to be* clashing *if its color differs from the color of the independent set to which it belongs. Denote by $\delta(G, \Phi)$ the following quantity:*

$$\delta(G, \Phi) = \textit{number of stones} + \textit{number of clashing stones}$$

The following result found by Demaine et al. motivates the introduction of this invariant.

Proposition 8. *Let Φ be a game position of Solitaire Clobber on a bipartite graph G. Then $\delta(G, \Phi) \bmod [3]$ keeps the same value during the game.*

Observe that if $k.n \equiv 0 \bmod [3]$ then the value of δ on a checkerboard position on a $k \times n$ grid is a multiple of 3. Nevertheless, a position consisting of a single stone has value 1 or 2 ; therefore when $k.n \equiv 0 \bmod [3]$, the $k \times n$ checkerboard is not 1-reducible.

Theorem 6 was proved for Partisan Solitaire Clobber and Proposition 8 asserts that one can not do better for the Impartial version.

Quasi-monochromatic positions for Impartial Solitaire Clobber.

One of the motivation to investigate the Impartial version of Solitaire Clobber is to consider game positions that are not balanced in the numbers of black and white stones. In this way, a natural game position is the quasi-monochromatic position. One may hope to solve this special position on a more general graph than a grid. Unfortunately, given a graph and a quasi-monochromatic position (with a unique white stone v), determining if this position is 1-reducible is equivalent to determine if G admits an Hamiltonian path starting from v. Therefore, the problem of 1-reducibility of a quasi-monochromatic position on a graph G is NP-complete.

In 1982, A. Itai, C. H. Papadimitriou, and J. L. Szwarcfiter [15] proved an even stronger result claiming that finding an Hamiltonian path in a grid graph is still NP-complete. They complete their result by a nice structural property of Hamiltonian paths in the $k \times n$ grid. Before stating their result we need to introduce some notations. Given a vertex A of the grid, we denote by $x(A)$ (respectively $y(A)$) the row (resp. column) of A.

Theorem 9 (A. Itai, C. H. Papadimitriou, and J. L. Szwarcfiter). *Let G be a $k \times n$ grid and A, B be two vertices of G. Let c be the 2-coloring of the grid. There exists an Hamiltonian path from A to B if and only if A and B satisfy the following conditions :*

- **Coloring condition:** *If $k.n$ is even then $c(A) \neq c(B)$. Otherwise $c(A) = c(B) = c(1,1)$.*
- **Connectivity condition:** *If $k = 1$ then A and B are the extremities of the grid. If $k = 2$ then either ($A = (1,1)$ and $B = (2,1)$) or ($A = (1,n)$ and $B = (2,n)$) or (A and B belong to distinct columns).*
- **Forbidden configurations:** *If $k = 3$, n is even and $c(A) \neq c(1,1)$ then if $x(A) = 2$ then $y(A) \geq x(B)$ otherwise $y(A) \geq y(B) - 1$ (see Figure 6).*

Figure 6: Forbidden configurations.

Observe that this last result solves the 1-reducibility of quasi-monochromatic positions on the $k \times n$ grid. Of course, one may provide a simpler proof since we need to fix only one extremity of the Hamiltonian path.

Corollary 10. *Let G be a $k \times n$ grid with $k \geq n \geq 2$, c be the 2-coloring of G and Φ be the quasi-monochromatic position with the unique white stone on v. Then (G, Φ) is 1-reducible if and only if either $c(v) = c(1,1)$ or $k.n$ is even.*

Proof. First remark that if $c(v) \neq c(1,1)$ and $k.n$ is odd then any pair $(A = v, B)$ does not satisfies the coloring condition of Theorem 9. Therefore there is no Hamiltonian path from A to B which implies that Φ is not 1-reducible.
Now, suppose that $c(v) = c(1,1)$ or $k.n$ is even. One may assume that $v \neq (1,1)$ and set $A = v$. If $k.n$ is even and $c(v) = c(1,1)$ then set $B = (2,1)$, otherwise $B = (1,1)$. Now, by Theorem 9, G admits an Hamiltonian path from A to B which gives a way to 1-reduce Φ. □

By Corollary 10, the only non 1-reducible positions are the ones for which $c(v) \neq c(1,1)$ and $k.n$ is odd. It is worth to notice that in that case, $\delta \equiv 0 \bmod [3]$.
Moreover, as proved in [11], in highest dimension $d \geq 3$, the coloring condition is sufficient to insure the existence of an Hamiltonian path from A to B in the d-dimensional grid. Therefore it is easier to play Impartial Solitaire Clobber on quasi-monochromatic positions in grids of large dimension.

Theorem 9 suggests to investigate the structural properties of spanning trees in the $k \times n$ grid. In [11], the authors generalized the Coloring and Connectivity conditions and add some forbidden configurations, in order to determine a necessary and sufficient condition on 3 vertices insuring that there exists a spanning tree of the grid having precisely these vertices as leaves. Nevertheless they were not able to solve the following general problem :

Problem 2. SPANNING TREE OF THE GRID WITH p FIXED LEAVES (p-STG)
Given a $k \times n$ grid G and p vertices $v_1, \dots, v_p$, does there exist a spanning tree of G having $v_1, \dots, v_p$ as leaves ?

Using an inductive proof based on the result of Itai et al. [15] (for $p = 2$), the authors in [11] proved that for any fixed integer $p \geq 2$, an analogue of the p-STG problem in grid graphs remains NP-Complete.

2.2 General Case

In this section, we will present a recent result due to Derouet-Jourdan and Gravier which solves Impartial Solitaire Clobber on "large enough" grids.

Theorem 11 (A. Derouet-Jourdan and S. Gravier). *Let Φ be a non-monochromatic position on a $k \times n$ grid with $k \geq n \geq 4$. Then Φ is 2-reducible.*

Sketch of Proof. Let Φ be a non-monochromatic position on a $k \times n$ grid G. Let $k = 4p_1 + r_1$ and $n = 4p_2 + r_2$ with $0 \leq r_i < 4$. Denote by R_i (respectively C_j) the restriction of (G, Φ) to the $i^t h$ row (resp. $j^t h$ column) of G. For $1 \leq i < p_1$ and $1 \leq j < p_2$, let

$$B(i,j) = (\cup_{s=i}^{i+3} R_s) \cap (\cup_{s=j}^{j+3} C_s).$$

For all $1 \leq i \leq p_1$, let

$$B(i,p_2) = (\cup_{s=i}^{i+3} R_s) \cap (\cup_{s=p_2}^{p_2+3+r_2} C_s).$$

Similarly, for all $1 \leq j \leq p_2$, let

$$B(p_1,j) = (\cup_{s=p_1}^{p_1+3+r_1} R_s) \cap (\cup_{s=j}^{j+3} C_s).$$

Finally, let

$$B(p_1,p_2) = (\cup_{s=p_1}^{p_1+3+r_1} R_s) \cap (\cup_{s=p_2}^{p_2+3+r_2} C_s).$$

Observe that the family $B(i,j)$ is a partition of G with $(p_1 - 1).(p_2 - 1)$ grids of size 4×4, (p_1-1) grids of size $(4+r_1)\times 4$, (p_2-1) grids of size $4\times(4+r_2)$ and one grid of size $(4+r_1)\times(4+r_2)$.

In the following of the proof, $\genfrac{}{}{0pt}{}{\bullet}{\circ}/\genfrac{}{}{0pt}{}{\circ}{\bullet}$ will denote indifferently one of the patterns $\genfrac{}{}{0pt}{}{\bullet}{\circ}$ or $\genfrac{}{}{0pt}{}{\circ}{\bullet}$.
We will say that a position on a $a \times b$ grid Q (with $a \geq b$) is *nice* if it can be reduced to one of the following patterns on the first column of Q :

- either $B_1 =$ 1 or 2 copies of $\genfrac{}{}{0pt}{}{\bullet}{\circ}/\genfrac{}{}{0pt}{}{\circ}{\bullet}$,
- or both ($B_2 =$ one copy of $\genfrac{}{}{0pt}{}{\bullet}{\circ}/\genfrac{}{}{0pt}{}{\circ}{\bullet}$ plus $\circ$ on some vertex v) and ($B_2' =$ one copy of $\genfrac{}{}{0pt}{}{\bullet}{\circ}/\genfrac{}{}{0pt}{}{\circ}{\bullet}$ plus $\bullet$ on v).

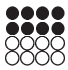
This property is interesting. Indeed, consider two consecutive blocks $B(i,j)$ and $B(i,j+1)$. We claim that if $B(i,j)$ is nice, then we can move the stones of B_t to $B(i,j+1)$ insuring that the resulting $B'(i,j+1)$ is non-monochromatic. To prove this claim, first assume that $B(i,j)$ can be reduced to B_1 with one copy of $\substack{\circ\\ \bullet}$:

$$\substack{\circ\\ \bullet}|\substack{\circ\\ \circ} \to \quad |\substack{\bullet\\ \circ}, \quad \substack{\circ\\ \bullet}|\substack{\bullet\\ \circ} \to \quad |\substack{\circ\\ \bullet}, \quad \substack{\circ\\ \bullet}|\substack{\bullet\\ \bullet} \to \quad |\substack{\bullet\\ \circ}, \quad \substack{\circ\\ \bullet}|\substack{\circ\\ \bullet} \to \quad |\substack{\bullet\\ \bullet} \text{ and } \quad |\substack{\circ\\ \circ}.$$

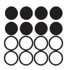
In the last case, we can choose either $\substack{\circ\\ \circ}$ or $\substack{\bullet\\ \bullet}$ in order to insure that the resulting $B'(i,j+1)$ is not monochromatic.
If $B(i,j)$ can be reduced to B_1 with two copies of $\substack{\bullet\\ \circ}/\substack{\circ\\ \bullet}$, then proceed as previously for each copy. Now, if $B(i,j)$ can be reduced to B_2 and B_2' then choose one of them according to the color of the stone placed on the neighbour of v in $B(i,j+1)$. For the copy of $\substack{\bullet\\ \circ}/\substack{\circ\\ \bullet}$ in B_2 or B_2' proceed as previously.

With the help of a computer to check a large number of cases, we proved the following lemma :

Lemma 12 (A. Derouet-Jourdan and S. Gravier). *Each non-monochromatic game position on a $4 \times n$ grid with $n = 4, \dots, 7$ is nice except the 4×4 position namely* Bad *depicted below :*



To settle the case of Bad's and monochromatic positions, we consider the concatenation of blocks. Firstly we consider the two following possible reductions of a Bad block:

 $\to \quad \substack{\bullet\\ \\ \bullet} \quad \text{and} \quad \substack{\circ\\ \\ \circ}$

Secondly, we check by computer that :

Lemma 13 (A. Derouet-Jourdan and S. Gravier). *For every game position on a 4×4 grid A, one of the following properties is true :*

$$\substack{\bullet\\ \\ \bullet} + A \textit{ is nice,} \quad \textit{or} \; \substack{\circ\\ \\ \circ} + A \textit{ is nice,} \quad \textit{or} \; \substack{\bullet\\ \\ \bullet} + A \;\to\; \substack{\bullet\\ \\ \bullet} \quad \textit{and} \quad \substack{\circ\\ \\ \circ}$$

To settle the monochromatic case, we explore again by computer a large number of cases :

Lemma 14 (A. Derouet-Jourdan and S. Gravier). *For every non white monochromatic position on a $4 \times n$ grid A with $n = 4, \dots, 7$, we have that A plus 2 white columns is nice.*

From Lemma 14, one can prove by induction that $M + A$ with M being a white monochromatic position on $4 \times p$ (with $p \geq 4$) and A being a non white monochromatic position on $4 \times q$ (with $4 \leq q \leq 7$), is nice.

Now to prove Theorem 11, consider an Hamiltonian path of blocks starting from $B(1,1)$ if p_1 is odd, otherwise from $B(1,p_2)$ and ending at $B(p_1,p_2)$. Applying Lemmas 12−14 along this path, we get a non-monochromatic position on the grid $k \times n$ with $4 \leq k \leq n \leq 7$. To achieve the proof, we check again by computer that every non-monochromatic position on a $k \times n$ grid, with $4 \leq k \leq n \leq 7$, is 2-reducible. $\square$

Since the Cartesian product of two (path) Hamiltonian graphs contains as a subgraph a 2-dimensional grid, one may mention the following corollary of Theorem 11.

Corollary 15. *Given two (path-)Hamiltonian graphs H and G of order at least 4, and Φ a non-monochromatic position on $G \Box H$, we have that Φ is 2-reducible.* $\square$

Remark that Corollary 15 implies "almost" the result on hypercubes (Theorem 26 appearing in Section 4). Nevertheless the proof is quite different, uses a computer to check an huge number of cases and claimed a weaker result.

The remaining questions left by Theorem 11 can be formulated as follows :

Problem 3. 2-Reducibility of small grids
Does there exist a non-monochromatic position on the $2 \times n$ or the $3 \times n$ grid (with $n \geq 2$) which is not 2-reducible ?

Our feeling is that, maybe, there exist such positions for the $2 \times n$ grid, but none for the $3 \times n$. The next problem is maybe easier :

Problem 4. Characterization of the 1-reducible grids
Given a non-monochromatic position Φ of the $k \times n$ grid (with $k \geq n \geq 4$), is Φ 1-reducible ?

A careful analysis of the proof in [6] may solve Problem 4. For the special positions studied previously (checkerboard and quasi-monochromatic), we remarked that if Φ is not 1-reducible then $\delta \equiv 0 \bmod [3]$. Therefore one may conjecture :

Conjecture 16. *A position Φ on a "large enough" grid G is not 1-reducible if and only if Φ is monochromatic or $\delta(G, \Phi) \equiv 0 \bmod [3]$.*

3 Algorithms on Impartial Solitaire Clobber

In this section, we only deal with Impartial Solitaire Clobber. We know that computing the reducibility value of a given game position is NP-hard in general. But for some specific cases, it can be done efficiently.

3.1 Paths and cycles

A game position on a path (with a stone on each vertex) will be considered as a word on the alphabet $\{\bullet, \circ\}$. We use the classical notations for words and languages: $\bullet^+$ represents all non-zero repetitions of $\bullet$, the notation $\bullet^\star$ represents the words $\bullet^+$ or the empty word. For example, $\bullet\circ^+\bullet$ defines all the game positions that begin with $\bullet$, continue with a sequence of $\circ$ (at least one), and end with $\bullet$.

We now give a characterization of the 1-reducible game positions on a path:

Lemma 17. *A 1-reducible path is either a single stone or in the form* $\bullet\circ^\star\bullet^\star\circ$*, or its symmetric* $\circ\bullet^\star\circ^\star\bullet$.

Proof. One easily sees that such paths are 1-reducible. We now prove that 1-reducible paths are necessarily in this form. First note that on a 1-reducible path, one must play from the extremities at each step. A different move would split the game into two subpaths, which would yield at least two stones in the end.
The proof is done by induction on the size of the path. Let P be a 1-reducible path of length strictly greater than 1. Without loss of generality, one may assume that $P = \bullet\circ P'$ and the first move gives $\bullet P'$. Since the length of $\bullet P'$ is less than the length of P, by induction, $\bullet P' = \bullet\circ^\star\bullet^\star\circ$ or P' is empty. Therefore $P = \bullet\circ\circ^\star\bullet^\star\circ$ or $P = \bullet\circ$. Both are in the form $\bullet\circ^\star\bullet^\star\circ$. $\square$

The set of words $\{\bullet\circ^\star\bullet^\star\circ, \circ\bullet^\star\circ^\star\bullet, \bullet, \circ\}$ is a regular language, which means that it can be recognized by a finite automaton. The automaton associated to 1-reducible paths is depicted by Figure 7. The initial state is 0, and the final ones are $1, 3, 5$. The set of 1-reducible paths can then be recognized in linear time.

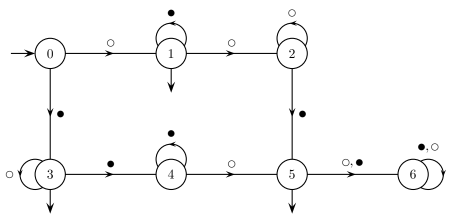


Figure 7: Automaton for 1-reducible paths.

Deciding whether a game position is k-reducible on a graph G consists in finding a partition of G into k 1-reducible subgraphs. We here give a way to see the reducibility value of a game position as the solution of a well-known problem in graph theory. When playing on paths and cycles, we show that the reducibility value can be computed in polynomial time by using this auxiliary problem.

Given a game position Φ on a graph G, consider the graph $H(G, \Phi)$ as the intersection graph of all 1-reducible subgraphs of (G, Φ) (i.e., the vertices of $H(G, \Phi)$ are the 1-reducible subgraphs of (G, Φ), and a pair of vertices (u, v) of $H(G, \Phi)$ is an edge iff the subgraphs u and v of G have a common vertex).

Proposition 18. *A game position* Φ *on a graph* G *is* k*-reducible if and only if* $H(G, \Phi)$ *admits a maximal independent set of cardinality* k.

Proof. Since Φ is k-reducible, there exists a partition $\mathcal{U} = \{U_1, ..., U_t\}$ of G into $t \leq k$ 1-reducible subgraphs. Each stone u of Φ must belong to exactly one 1-reducible subgraph U_i of the partition. This means that in $H(G, \Phi)$, $\mathcal{U}$ must be independent (at most one U_i per stone) and maximal (each stone must be covered by at least one U_i).

Let $\mathcal{U} = \{U_1, ..., U_k\}$ be a maximal independent set of $H(G, \Phi)$. Since $\mathcal{U}$ is independent and each U_i is a 1-reducible subgraph, then the restriction of Φ to $\cup U_i$ is k-reducible. Now since each vertex of G is a vertex of $H(G, \Phi)$, by maximality of $\mathcal{U}$, the position Φ is k-reducible. □

In terms of graph theory, the maximal independent set problem is known as:

Independent Dominating Set
Given an integer k and a graph G, does there exist an independent set D of G of cardinality $\leq k$ such that each vertex v of G either belongs to D or is adjacent to a vertex of D ?

This problem is NP-complete for arbitrary graphs (see [10, 4, 13]). R. Irving [14] showed that even the problem of finding an independent dominating set within a factor of t of a smallest independent dominating set is still NP-hard. According to Proposition 18, it will be interesting to consider the following problem:

Problem 5. Approximating scheme for Solitaire Clobber
Given a game position Φ on a graph, does there exist an integer t and an algorithm reducing Φ to a number of stones within a factor of t of the reducibility value of Φ ?

Nevertheless Independent Dominating Set is known to be polynomially solvable for some classes of graphs. For instance, Klostermeyer et al. [16] proposed a quadratic time algorithm for arc-circular graphs.
The class of arc-circular graphs contains the intersection graphs of paths (known as *interval graphs*) and cycles. Moreover, since the number of 1-reducible subgraphs of a path (resp. cycle) on n vertices is at most n^2, one may use the algorithm of Klostermeyer et al. to determine the reducibility value of paths (resp. cycles) in $O(n^4)$ time. However, Blondel et al. suggest in [2] another approach to compute this value in linear time. They encode a position by words, but letters refer to separations between stones: s means that it is a separation between two stones with same color and a means it is a separation between opposite colors. For example, the position ●●●○○●○○● is represented by the word *ssasasa*. This breaks the symmetry between the colors since each word encodes two opposite positions.

Theorem 19 (Blondel, de Kerchove, Hendrickx and Jungers). *Given a game position Φ on a path on n vertices, the reducibility value of Φ can be computed in $O(n)$ time.*

By removing an edge of a cycle, we get a path and we can compute its reducibility value in linear time. This provides an algorithm running in quadratic time. Blondel et al. [2] improve this complexity to linear time.

Corollary 20 (Blondel, de Kerchove, Hendrickx and Jungers). *The reducibility value of a game position on a cycle can be computed in linear time.*

Proof. Let $v_1, \ldots, v_n$ be the n vertices of a cycle C of size n with edges (v_i, v_{i+1}) where the subscript is taken modulo n. Let Φ be a non-monochromatic position on C. Suppose that $\Phi(v_1) = \bullet$. We consider all the different ways of playing v_1. There are five cases.

- No move involves v_1. See (C) in Figure 8. Then $rv(C, \Phi) \leq rv(C - v_1, \Phi) + 1$.

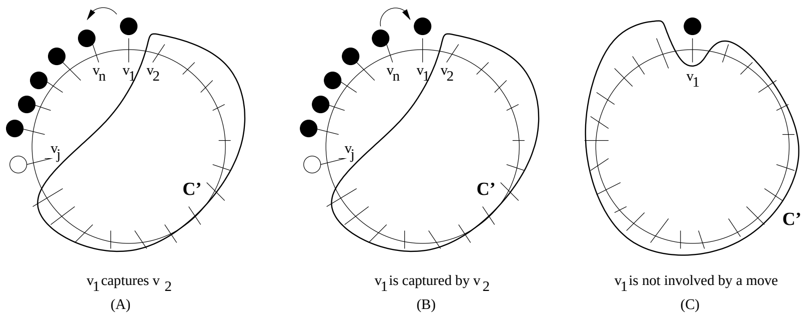


Figure 8: Possible moves for v_1.

- The vertex v_1 is clobbered by v_n. See (B) in Figure 8. This is possible only if the first white vertex v_j at the left of v_1 moves to v_n. Let C' be the path induced by $v_1, \ldots, v_{j-1}$ and Φ' be a position on C' defined by $\Phi'(v_1) = \circ$ and $\Phi'(v_i) = \Phi(v_i)$ for all $i = 2, \ldots, j-1$. Therefore, $rv(C, \Phi) \leq rv(C', \Phi')$.
- The vertex v_1 is clobbered by v_2. This case is similar to the previous one.
- The vertex v_1 clobbers v_n. See (A) in Figure 8. This is possible only if the first white vertex v_j at the left of v_1 moves to v_n. If $j = n$, let $C' = C - v_1$ and Φ' be a position on C' such that $\Phi'(v_i) = \Phi(v_i)$ for all $i \neq 1, n$ and $\Phi'(v_n) = \bullet$. Then $rv(C, \Phi) \leq rv(C', \Phi')$. Otherwise let C' be the path induced by $v_2, \ldots, v_{j-1}$ and Φ' be the restriction of Φ on C'. We thus have $rv(C, \Phi) \leq rv(C', \Phi') + 1$.
- The vertex v_1 clobbers v_2. This case is similar to the previous one.

Now, since we explored all the possibilities to play v_1, we get $rv(C, \Phi)$ by taking the minimum value among all the five possibilities. Thanks to Theorem 19, we obtain a linear time algorithm to compute $rv(C, \Phi)$. □

3.2 Trees

In [9], Farber proved that the Independent Dominating Set problem is polynomial on chordal graphs. A graph G is chordal iff there exist a tree T and a family $\mathcal{F}$ of subtrees of T such that G is the intersection graph of $\mathcal{F}$. Therefore, one may use Proposition 18 in order to find a polynomial time algorithm for Impartial Solitaire Clobber on trees. Unfortunately, given a tree T and a game position on T we don't know whether the number of 1-reducible induced subtrees of T is polynomial.

However, deciding whether a game position on a tree is 1-reducible can be computed in linear time: it suffices to play recursively from the hanging vertices whenever it is possible (a hanging vertex is a vertex of degree 1). If there remains a unique stone in the end, then the tree is 1-reducible. Otherwise, it is not.

In this section, we propose a dynamic algorithm to solve Impartial Solitaire Clobber on trees. This result was obtained in collaboration with Luerbio Faria from State University of Rio de Janeiro.

Theorem 21. *Given a position Φ on a tree $T = (V, E)$, the value $rv(T, \Phi)$ can be computed polynomially in $O(|V|^3)$.*

Let Φ be a position on the tree $T = (V, E)$, and let r be any vertex of T. We will consider a dynamic algorithm on T rooted in r. For each vertex $s \in V$, denote by $T(s)$ the subtree of T induced by s and all its children (i.e., from s to the leaves). Clearly $T(r)$ is equal to T itself. Denote by $J = \{s_1, \dots, s_p\}$ the set of the children of s.

For all vertex s, there are three possibilities in a way of playing in $T(s)$:

- in the end s is $\circ$.
- in the end s is $\bullet$.
- in the end s supports no stone.

Remark that in the third case, the stone on s moves to some vertex s_i at some step during the game. Therefore it corresponds to playing a *joker* move in the subtree $T(s_i)$.

Hence for each vertex s of T scanned from the leaves to the root, we compute five *k-values*:

1. $k_{-1}(s)$, which is the reducibility value of $T(s)$ when a stone $\circ$ is left on s in the end.
2. $k_1(s)$, which is the reducibility value of $T(s)$ when a stone $\bullet$ is left on s in the end.
3. $k_0(s)$, which is the reducibility value of $T(s)$.
4. $k_{bj}(s)$, which is the reducibility value of $T(s)$ when the *black joker* is played. The *black joker* consists in turning a white stone $\circ$ on s into a black one $\bullet$ whenever during the play. It must be used exactly once.
5. $k_{wj}(s)$, which is the reducibility value of $T(s)$ when the *white joker* is played. The *white joker* consists in turning a black stone $\bullet$ on s into a white one $\circ$ whenever during the play. It must be used exactly once.

When computing a k-value, if $T(s)$ is monochromatic $\circ$ (resp. $\bullet$), then we assign the value ∞ to $k_1(s)$ and $k_{wj}(s)$ (resp. $k_{-1}(s)$ and $k_{bj}(s)$).

Note that $k_0(r)$ is exactly the value $rv(T, \Phi)$. Then we also have for all vertex s, $k_0(s) \leq \min(k_{-1}(s), k_1(s))$.

Now consider a way of playing on $T(s)$ such that there is no move from s to some s_i. Let

$$J_{-1} = \{s_i \mid \text{there is a move of a stone } \circ \text{ from } s_i \text{ to } s\}$$

$$J_1 = \{s_i \mid \text{there is a move of a stone} \bullet \text{ from } s_i \text{ to } s\}$$

$$J_0 = \{s_i \mid \text{there is no move between } s_i \text{ to } s\}$$

Remark that by definition of J_1 and J_{-1}, we have

$$-1 \leq |J_1| - |J_{-1}| \leq 1.$$

The value of this way of playing (i.e., the number of remaining stones) is thus greater or equal to

$$W(J_{-1}, J_1, J_0) = \sum_{i \in J_{-1}} (k_{-1}(s_i) - 1) + \sum_{i \in J_1} (k_1(s_i) - 1) + \sum_{i \in J_0} k_0(s_i) + 1.$$

A part of the resolution of Impartial Solitaire Clobber on trees relies on the following problem:

J-TRIPARTITION
Given an integer a, a set $J = \{s_1, \ldots, s_p\}$ and three mappings k_i (with $i \in \{-1, 0, 1\}$) from J to $\mathbb{N} \cup \{\infty\}$, find a partition $J_{-1} \cup J_0 \cup J_1$ of J which minimizes

$$W = \textstyle\sum_{i \in J_{-1}} (k_{-1}(s_i) - 1) + \sum_{i \in J_1} (k_1(s_i) - 1) + \sum_{i \in J_0} k_0(s_i) + 1$$

under the following constraint:

$$|J_1| - |J_{-1}| = a \qquad (1)$$

Lemma 22. J-TRIPARTITION *can be solved in* $O(|J|^2)$ *time.*

Proof. For all $s_i \in J$, set $x_i = k_1(s_i) - k_0(s_i) - 1$ and $y_i = k_{-1}(s_i) - k_0(s_i) - 1$. Let $X = \{i \mid s_i \in J_1\}$ and $Y = \{i \mid s_i \in J_{-1}\}$. Thus, x_i represents the relative gain of putting s_i in J_1 and y_i the relative gain of putting s_i in J_{-1}. The overall gain is then $W(X, Y) = \sum_{i \in X} x_i + \sum_{i \in Y} y_i$. We want to minimize this quantity with respect to the constraint $|X| - |Y| = a$. Without loss of generality, assume that $a \geq 0$.

Additionally one may assume that $a \geq p$, otherwise there is no solution.

Let (X_n^*, Y_n^*) be an optimal solution, i.e., a solution such that $W(X_n^*, Y_n^*)$ is minimum with respect to the fact that $|X_n^*| = n$ and $|X_n^*| - |Y_n^*| = a$.

Let (X_n, Y_n) be the solution produced by Algorithm 1 with $|X_n| = n$.
If $n = a$, then clearly $W(X_n, Y_n) = W(X_n^*, Y_n^*)$. Now suppose $n > a$, and assume $W(X_k, Y_k) = W(X_k^*, Y_k^*)$ for all $k < n$, where the pair (X_k, Y_k) is obtained by Algorithm 1 and (X_k^*, Y_k^*) are two optimal sets with respect to the fact that $|X_k^*| = k$ and $|X_k^*| - |Y_k^*| = a$.

Let Δ denote the set $(X_n^* \cup Y_n^*) \setminus (X_{n-1} \cup Y_{n-1})$. Since $|X_n^*| > |X_{n-1}|$ and $|Y_n^*| > |Y_{n-1}|$, we have $|\Delta| \geq 2$.

Algorithm 1 J-TRIPARTITION

Require: (x_i, y_i) for $i \in \{1, 2, \dots, p\}$ and a

Ensure: $\sum_{i \in X^*} x_i + \sum_{i \in Y^*} y_i$ is minimum and $|X^*| - |Y^*| = a$

```
put the a elements i with the smallest values of x_i in X_a and set Y_a = ∅
n ← a

repeat
  X* ← X_n
  Y* ← Y_n
  n ← n + 1
  For all triples (i', j', j), take the minimum value of W(X_n, Y_n) among the
  three following cases:

  1.  For i', j' ∉ {X_{n-1}, Y_{n-1}}, set X_n = X_{n-1} ∪ {i'}, Y_n = Y_{n-1} ∪ {j'}.

  2.  For i', j' ∉ {X_{n-1}, Y_{n-1}} and j ∈ Y_{n-1}, set X_n = X_{n-1} ∪ {j}, Y_n = (Y_{n-1} \ {j}) ∪
      {i', j'}.

  3.  For i', j' ∉ {X_{n-1}, Y_{n-1}} and j ∈ X_{n-1}, set Y_n = Y_{n-1} ∪ {j}, X_n = (X_{n-1} \ {j}) ∪
      {i', j'}.

until (W(X_n, Y_n) ≥ W(X*, Y*) or (n > a + ⌊(p-a)/2⌋))

if W(X_n, Y_n) < W(X*, Y*) then
  X* ← X_n
  Y* ← Y_n
end if
```

Case I: There exist $i, j \in \Delta$ such that $i \in X_n^*$ and $j \in Y_n^*$.
By induction hypothesis, we have $W(X_n^* \setminus \{i\}, Y_n^* \setminus \{j\}) \geq W(X_{n-1}, Y_{n-1})$. Moreover, by the first item of Algorithm 1 we get $W(X_{n-1} \cup \{i\}, Y_{n-1} \cup \{j\}) \geq W(X_n, Y_n)$. Thus $W(X_n^*, Y_n^*) \geq W(X_n, Y_n)$.

Case II: $\Delta \subset X_n^*$.
Let i, j be two distinct elements of Δ. Then, since $|Y_n^*| > |Y_{n-1}|$, there exists $l \in Y_n^* \setminus Y_{n-1}$. As $\Delta \subset X_n^*$, it turns out that $l \notin \Delta$ and thus $l \in X_{n-1}$. By induction hypothesis we have $W(X_n^* \cup \{l\} \setminus \{i, j\}, Y_n^* \setminus \{l\}) \geq W(X_{n-1}, Y_{n-1})$. And by the second item of Algorithm 1 we obtain $W(X_{n-1} \cup \{i, j\} \setminus \{l\}, Y_{n-1} \cup \{l\}) \geq W(X_n, Y_n)$. Thus $W(X_n^*, Y_n^*) \geq W(X_n, Y_n)$.

Case III: $\Delta \subset Y_n^*$.
Symmetric to case II, by considering the third item of Algorithm 1.

Complexity analysis:
The initialization step can be done in $O(p \log(p))$. In the loop, the considered sets need at most

$O(p^3)$ operations to be enumerated. The update of best sets costs $O(1)$. In the end, we get a $O(p^4)$ time algorithm.

However, in Algorithm 1 we do not need to consider all the possible triples (i', j', j), but only one that minimizes W. To do this, we start by sorting the sets $A = \{x_1, \dots, x_p\}$, $B = \{y_1, \dots, y_p\}$ and $C = \{x_1 - y_1, \dots, x_p - y_p\}$ by increasing order. This can be done in $O(p \log p)$ time. Additionally consider the function $\xi : \{1, \dots, p\} \to \{x, y, z\}$ such that

$$\xi(i) = \begin{cases} x & \text{if } i \in X_{n-1} \\ y & \text{if } i \in Y_{n-1} \\ z & \text{otherwise} \end{cases}$$

Let

$$\begin{aligned} i'_1 &= \text{index of } \min\{x_i \,|\, \xi(i) = z\} \\ j'_1 &= \text{index of } \min\{y_j \,|\, \xi(j) = z \text{ and } j \neq i'_1\} \\ i'_2 &= \text{index of } \min\{x_i \,|\, \xi(i) = z \text{ and } i \neq i'_1\} \\ j_1 &= \text{index of } \min\{x_i - y_i \,|\, \xi(i) = y\} \\ j_2 &= \text{index of } \max\{x_i - y_i \,|\, \xi(i) = x\} \end{aligned}$$

Each one of the indices above can be computed in $O(p)$ time thanks to the sorts of A, B and C. Now for each step in the loop of Algorithm 1, it is sufficient to consider the triple (i', j', j) such that $(i', j') = (i'_1, j'_1)$ or (i'_2, i'_1). For the step 2, set $j = j_1$, and for the step 3, set $j = j_2$. Finally, since updating ξ can be done in constant time, we get a $O(p^2)$ time algorithm. □

We now have enough ingredients to prove Theorem 21.

Proof of theorem 21: By induction from the leaves to the root r of T.
Let l be a leaf of T. We have straightforwardly $k_0(l) = 1$. If l is white, then $k_{-1}(l) = k_{bj} = 1$ and $k_1(l) = k_{wj} = \infty$. Conversely if l is black.

We now suppose that for some vertex s of T, the five k-values of each child s_i for $1 \leq i \leq p$ are known. Then the k-values of s can be computed as follows:

1. $k_{-1}(s)$. Since s must be white in the end, there is no move from s to some s_i. Thus an optimal way of playing on $T(s)$ is a solution of J-Tripartition with $a = 0$ if s is initially ○, otherwise $a = -1$.
2. $k_1(s)$. Identical to the previous case with $a = 0$ or 1 according to the initial color of s.
3. $k_0(s)$. Let us consider first the computation of $k_\emptyset(s)$, which is the reducibility value of $T(s)$ when there remains no stone on s in the end. If there remains no stone on s in the end, one must play from s to some child s_j. Assume that s_j is fixed. There are two possible ways of playing to compute $k_\emptyset(s)$ (depending on whether $s \to s_j$ is a black or white move): as for $k_1(s)$, compute the reducibility value of $T(s) \setminus T(s_j)$ when a stone ● is left on s in the end. Denote by W_b this value. Play then on $T(s_j)$ by considering the $s \to s_j$ move as a black joker. These operations leave $W_b + k_{bj}(s_j) - 1$ stones on $T(s)$. Do the same by yielding a stone ○ on s and then considering the white joker.

Apply this method by fixing each child $s_j \in J$, and take the minimum value among these $2|J|$ computations.

Now set $k_0(s) = \min(k_{-1}(s), k_1(s), k_\emptyset(s))$.

4. $k_{bj}(s)$. Proceed as for the computation of $k_0(s)$. But since we have one additional move (the joker one $\bullet \to \circ$), we need an additional child in J_{-1}. Hence replace a by $a-1$ in the previous cases.

5. $k_{wj}(s)$. Identical to the computation of $k_{bj}(s)$, except that we replace a by $a+1$ in the previous cases.

When considering the complexity of this algorithm, each vertex is scanned exactly once to compute its five k-values. The computation of $k_0(s)$ is the highest one, since it requires to apply $|J|$ times the J-TRIPARTITION algorithm. Therefore $k_0(s)$ needs $O(p^3)$ operations. Since p is at most the degree of a vertex, we get a total complexity in $O(n^3)$. □

Remark that if we apply the algorithm of Theorem 21 for a path where the root is a leaf, then $p = 1$ at each time we apply J-TRIPARTION. Hence this algorithm also solves IMPARTIAL SOLITAIRE CLOBBER on paths in linear time.

It will be interesting to develop algorithms for other classes of graphs. Since the structure of chordal graphs is closed to the one of trees, one may ask the following problem:

Problem 6. IMPARTIAL SOLITAIRE CLOBBER ON CHORDAL GRAPHS
Given a position Φ on a chordal graph G, can we compute $rv(G, \Phi)$ in polynomial time ?

4 Other classes of graphs

In this section we determine the reducibility value for some classes of graphs.

4.1 Impartial Solitaire Clobber on Hamming graphs

Playing Solitaire Clobber on cliques is not a tricky activity. It is straightforward that all non-monochromatic cliques are 1-reducible. A more detailed analysis even lead to the fact that one can choose the color and the location of the final stone.

Hamming graphs are multiple Cartesian product of cliques. The clique K_n, $K_2 \Box K_3$ or $K_4 \Box K_5 \Box K_2$ are examples of Hamming graphs. Hypercubes, defined by $Q_n = \Box^n K_2$, constitute the most famous class of Hamming graphs. It is well-known that any Hamming graph admit an Hamiltonian cycle.

For these reasons, it is not surprising that the reducibility value of a position on a Hamming graph is small. From Corollary 15, it turns out that Hamming graphs are 2-reducible. Note that this result was already contained in [7], where the case of Hamming graphs is fully investigated.

As for the case of cliques, one can reduce "strongly" the family of Hamming graphs, in the sense that one can choose the location and the color of the final stones. This property is developed through the two following definitions:

Definition 23. *A graph G is strongly 1-reducible if: for any vertex v of G, for any position Φ on G (provided $(G \setminus v, \Phi)$ is not monochromatic), for any color c ($\bullet$ or $\circ$), there exists a way of playing that yields a unique stone of color c on v.*

Definition 24. *A graph G is strongly 2-reducible if: for any vertex v of G, for any position Φ on G (provided $(G \setminus v, \Phi)$ is not monochromatic), for any two colors c and c' (provided there exist two different vertices u and u' such that $\Phi(u) = c$ and $\Phi(u') = c'$), there exists a way of playing that yields a stone of color c on v, and (possibly) a second stone of color c' somewhere else.*

The following result asserts that almost all the Hamming graphs are strongly 1-reducible.

Theorem 25 (Dorbec, Duchêne, Gravier). *Any Hamming graph that is neither $K_2 \Box K_3$ nor an hypercube is strongly 1-reducible.*

The case of hypercubes and the graph $K_2 \Box K_3$ are a bit different.
In the case of hypercubes, there are some game positions for which the reducibility value equals 2. For example, the position $\substack{\bullet\circ \\ \bullet\circ}$ on the hypercube Q_2 is not 1-reducible. Dorbec et al. showed in [7] that for all integer $n > 1$, there exists a non-monochromatic position on Q_n which is not 1-reducible. Nevertheless, hypercubes were proved to be 2-reducible, and even more since the location and the color of the final stones can be fixed (under some conditions). It corresponds to Theorem 26.

Theorem 26 (Dorbec, Duchêne, Gravier). *Hypercubes are strongly 2-reducible.*

Note that this result is not sufficient to quickly decide whether or not a game position on an hypercube is 1-reducible. In [7] Conjecture 27 is proposed. Since hypercubes are bipartite graphs, the invariant δ of Definition 7 and Proposition 8 is available.

Conjecture 27. *The non-monochromatic game positions Φ on the hypercube Q_n that are 1-reducible are exactly those for which the invariant $\delta(Q_n, \Phi) \not\equiv 0 \bmod [3]$.*

We conclude this part with the Hamming graph $K_2 \Box K_3$. It was proved in [7] that non-monochromatic positions on this graph are always 1-reducible. However the conditions for strong 1-reducibility are not fulfilled.
One may ask whether there are strong 1-reducible graphs which are not Hamming graphs, and more generally :

Problem 7. STRONGLY 1-REDUCIBLE GRAPHS
Characterize all the graphs that are strongly 1-reducible.

4.2 Impartial Solitaire Clobber on complete multipartite graphs

Given $k \geq 2$ integers $n_1, \ldots, n_k$, the complete k-partite graph $K_{n_1,\ldots,n_k}$ is the graph $G = (V, E)$ such that V can be partitioned into k independent sets S_i with $|S_i| = n_i$ for $1 \leq i \leq k$, and for all $u \in S_i$ and $v \in S_j$ with $i \neq j$ we have $(u, v) \in E$. See Figure 9 for an example.

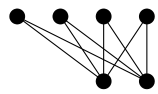


Figure 9: The complete bipartite graph $K_{4,2}$.

For convenience, the edges of the complete k-partite graphs will not be drawn in the other figures.

Complete bipartite graphs

As shown in the previous sections, the invariant $\delta(G, \Phi)$ defined by Demaine et al. is a powerful tool to compute the reducibility value on several families of bipartite graphs. It is also the case of balanced complete bipartite graphs $K_{n,n}$, as proved in [8]. For such graphs we have a result similar to the one of checkerboard grids:

Proposition 28 (Duchêne, Gravier, Moncel). *A position Φ on a balanced complete bipartite graph $K_{n,n}$ is 1-reducible if $\delta(K_{n,n}, \Phi) \not\equiv 0 \bmod [3]$. Otherwise, the reducibility value is equal to 2.*

When the complete bipartite graph is unbalanced, the reducibility value may be higher. The invariant δ is thus not sufficient to compute the exact reducibility value.

Let us denote by $K_{n,m}$ with $n > m > 0$ an unbalanced complete bipartite graph, and Φ a game position on $K_{n,m}$.
The values n_b and n_w denote respectively the numbers of black and white stones in the independent set of size n. Obviously we have $n = n_b + n_w$. Similarly, m_b and m_w denote respectively the numbers of black and white stones in the independent set of size m.
Without loss of generality, we consider positions satisfying $n_b \leq n_w$. Under this condition, we define a nonnegative integer $q = n - 2n_b$. We also define the function $f(\Phi) = q - m - m_b$.

Theorem 29 (Duchêne, Gravier, Moncel). *Let Φ be a game position on $K_{n,m}$ with $n > m > 0$ and $n_b \leq n_w$.*
If $f(\Phi) < 0$ then $rv(K_{n,m}, \Phi) \leq 2$ and $rv(K_{n,m}, \Phi) = 1$ iff $\delta(K_{n,m}, \Phi) \not\equiv 0 \bmod [3]$.
If $f(\Phi) \geq 0$ then $rv(K_{n,m}, \Phi) = f(\Phi) + 2$.

The ground of Theorem 29 can be explained as follows. When $f(\Phi) < 0$, it is always possible to reduce Φ to a non-monochromatic position on a balanced complete bipartite graph $K_{m,m}$. This operation can be done by alternating black and white moves to reduce the size of the larger independent set, or by clobbering at most $m_b - 1$ black stones of the smaller independent set. Two examples of such positions are given by Figure 10. Once the graph is reduced to a balanced $K_{m,m}$, it suffices to use Proposition 28.

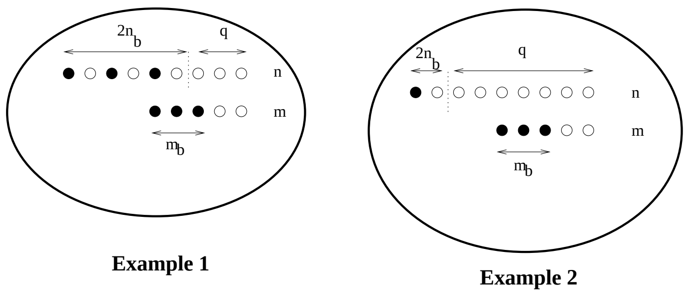


Figure 10: Two examples of positions satisfying $f(\Phi) < 0$

When $f(\Phi) \geq 0$, it means that the number of white stones in the larger independent set is too big comparing to the number of black stones in the smaller independent set. A reduction to a balanced position is not possible. Figure 11 illustrates this situation. In such cases, the reducibility value equals $f(\Phi) + 2$. One key of the proof is to show that when playing any move from Φ, the function f never decreases.

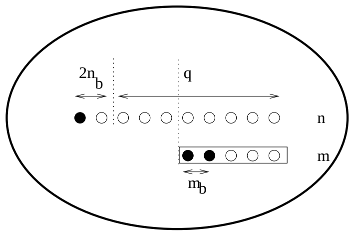


Figure 11: A position satisfying $f(\Phi) \geq 0$

Complete k-partite graphs with $k \geq 3$

In the case of complete k-partite graphs with $k \geq 3$, the invariant δ is no more available. Nevertheless, the introduction of the function f for complete bipartite graphs is the key for the other k-partite graphs. We distinguish two cases, depending on the number of maximum independent sets of the graph.

Theorem 30 (Duchêne, Gravier, Moncel). *Let Φ be a non-monochromatic position on a k-partite graph G with $k \geq 3$. If G has at least two independent sets of maximum size, then Φ is* 1*-reducible.*

An example of such a 1-reducible position is given by Figure 12.

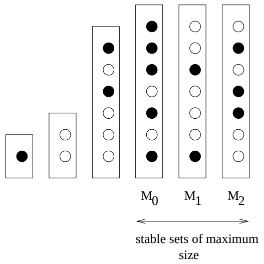


Figure 12: 6-partite complete graph with three maximum independent sets

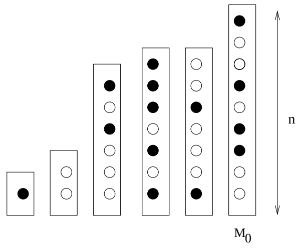


Figure 13: 6-partite complete graph with a unique maximum independent set

When the k-partite graph G has a unique maximum independent set M_0 (as on Figure 13), then the reducibility is once again fixed by the function f. The parameters introduced for complete bipartite graphs are extended for k-partite graphs : the value n is the size of the maximum independent set M_0, the quantities n_b and n_w are the numbers of black and white stones in M_0, the parameter m is the size of $G \setminus M_0$, and q is still equal to $n - 2n_b$. The function f is unchanged (i.e., equal to $q - m - m_b$). Then the reducibility value is similar as in Theorem 29 :

Theorem 31 (Duchêne, Gravier, Moncel). *Let Φ be a non-monochromatic position on a k-partite graph G with $k \geq 3$. If $f(\Phi) < 0$ then Φ is 1-reducible. If $f(\Phi) \geq 0$ then $rv(K_{n,m}, \Phi) = f(\Phi) + 2$.*

Problem 8. General invariant
In view of the previous result and the use of the function f, could we provide a general invariant that extends δ to non-bipartite graphs ?

5 General results about Impartial Solitaire Clobber

If you play Impartial Solitaire Clobber on a random graph with a random position, you have great chance to leave a unique stone in the end. This result is due to Ruszinkó [18] and can be formulated in the following way in terms of random graphs:

Proposition 32 (Ruszinkó). *Almost all graphs are* 1*-reducible.*

In other words, it means that for any fixed $p \in]0, 1[$, the random graph $G_{n,p}$ is 1-reducible with probability tending to 1 as n tends to infinity – for details about Erdős-Rényi's random graph model $G_{n,p}$, see [3].

In contrast with this probabilistic result, the study of extremal game positions of Impartial Solitaire Clobber was investigated in [8, 2]. The following part deal with the main results relating to this topic.

Given a graph G, let us denote by $maxrv(G) = \max(rv(G, \Phi))$ taken over all non-monochromatic game positions Φ on G, and $minrv(G)$ the minimum of these values taken over the same set. In other words, $maxrv(G)$ is the worst arrangement of the stones that can be found on G, and

$minrv(G)$ is the the one that minimizes the reducibility value.

In [8], the authors tried to estimate the values of $minrv(G)$ and $maxrv(G)$ for all graph G. The first result asserts that it is always possible to find a 1-reducible position on any connected graph.

Proposition 33. *Let G be a connected graph. Then we have $minrv(G) = 1$.*

Proof. Let $G = (V, E)$ be a connected graph. We will prove a stronger result : for any vertex u of G and any color $c \in \{\circ, \bullet\}$, there exists a position Φ on G such that (G, Φ) is 1-reducible with the last stone located on u and with color c.
The proof works by induction. It is obvious if $|V| = 1$. Now suppose that $|V| \geq 2$. Let u be a vertex of G and $c \in \{\bullet, \circ\}$. Let $G_1, \ldots, G_t$ be the t connected components of $G - u$. For each $i = 1, \ldots, t$, let u_i be a neighbour of u in G_i. Without loss of generality, assume that $c = \bullet$. Set $c_i = \bullet$ for $1 \leq i \leq \lceil t/2 \rceil$, and $c_j = \circ$ for $\lceil t/2 \rceil < j \leq t$. By induction hypothesis, there exists a position Φ_i on G_i such that there is a way of playing that yields a unique stone of color c_i on u_i. Now consider the position Φ on G defined as follows: $\Phi(v) = \Phi(u)$ for all $v \neq u$, $\Phi(u) = \bullet$ if t is even, and $\circ$ otherwise. If t is even, then play alternately a pair $(u_j \to u, u_i \to u)$ with $i \leq \lceil t/2 \rceil$ and $j > \lceil t/2 \rceil$. If t is odd, play first $u_1 \to u$ and then as the case where t is even. $\square$

The discussion is more tricky when estimating the value of $maxrv(G)$. Upper bounds are given in [8]. They involve the minimum degree d of the graph.

Theorem 34 (Duchêne, Gravier, Moncel). *Given a graph G of minimum degree d, we have $maxrv(G) \leq n - d$.*

Theorem 34 asserts that it is always possible to find a strategy with at least d moves. Note that a greedy strategy is provided to do so.

When d is fixed, Duchêne et al. give in [8] a characterization of the graphs G satisfying exactly $maxrv(G) = n - d$. Roughly speaking, they correspond to stars of cliques of size $d + 1$. This family of graphs is fully defined below and is denoted by $\mathcal{G}_d$.

For any $d \geq 1$, let us define $\mathcal{G}_d$, a set of connected graphs of degree minimum d as follows:

- the complete graph on $d + 1$ vertices belongs to $\mathcal{G}_d$ for all $d \geq 1$.
- for any integer $k \geq 2$, let us define $S_k(K_{d+1})$ as the graph obtained by k disjoined copies of K_d plus one additional vertex v adjacent to all the vertices of each K_d.
- for $d = 2$, add the cycle on 4 vertices C_4.
- no other graph belongs to $\mathcal{G}_d$.

Theorem 35 (Duchêne, Gravier, Moncel). *For all $d \geq 1$, the set of connected graphs G having minimum degree d and such that $maxrv(G) = n - d$ is exactly $\mathcal{G}_d$.*

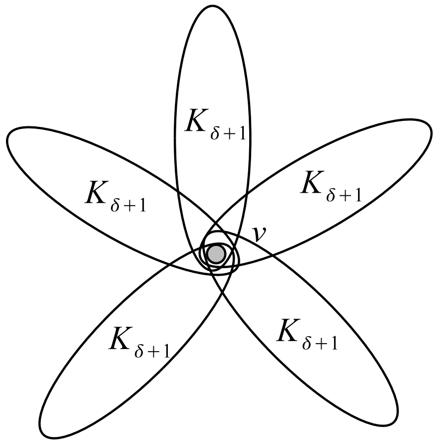


Figure 14: The graph $S_k(K_{d+1})$.

For example, a position Φ on a graph $G \in \mathcal{G}_d$ that satisfies $rv(G, \Phi) = n - d$ can be built as follows :

- If $G = K_{d+1}$, then choose any non-monochromatic position Φ on G.
- If $G = S_k(K_{d+1})$, then choose $\Phi(v) \neq \Phi(u)$ for all vertex $u \neq v$.
- If $G = C_4 = (v_1, \dots, v_4)$, then choose a position Φ such that $\Phi(v_1) = \Phi(v_2) = \bullet$ and $\Phi(v_3) = \Phi(v_4) = \circ$.

Given a graph G, it seems that the computation of $maxrv(G)$ is a hard problem.

Problem 9. MAXRV(G)
Given a graph G and an integer k, does there exist a polynomial time algorithm deciding whether $maxrv(G) \leq k$?

Blondel et al. give in [2] a closed formula for $maxrv(G)$ when G is a path or a cycle. Their results are summed up below:

Proposition 36 (Blondel, Kerchove, Hendrickx and Jungers). *If P is a path of size n, then $maxrv(P) = \lceil n/2 \rceil$.*

Note that on a path P of size n, examples of game positions Φ satisfying $rv(P, \Phi) = \lceil n/2 \rceil$ are those in the form $\bullet^k \circ \bullet^k$.

Proposition 37 (Blondel, Kerchove, Hendrickx and Jungers). *If C is a cycle of size n, then $maxrv(C) = \lceil n/3 \rceil$.*

This value is reached for positions consisting in repetitions of the pattern $(\bullet\bullet\circ)$.

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